CLASSIFYING COMPLEMENTS FOR HOPF ALGEBRAS AND LIE ALGEBRAS

A. L. AGORE AND G. MILITARU

Abstract. Let $\mathcal{C}$ be the category of Lie algebras or Hopf algebras and $A \subseteq E$ a given extension in $\mathcal{C}$. We answer the *classifying complements problem* (CCP) which consists of the explicit description and classification of all complements of A in E. We prove that if H is a given complement then all the other complements are obtained from H by a certain type of deformation which we introduce for the categories above. We establish a bijective correspondence between the isomorphism classes of all complements of A in E and a cohomological type object denoted by $\mathcal{HA}^2(H, A \,|\, (\triangleright, \triangleleft))$, where $(\triangleright, \triangleleft)$ is the canonical matched pair associated to H. The factorization index $[E : A]^f$ is introduced as a numerical measure of the (CCP): it is the cardinal of the isomorphism classes of all complements of A in E. For two n-th roots of unity we construct a $4n^2$-dimensional quantum group whose factorization index over the group algebra $k[C_n]$ is arbitrary large. The answer to the (CCP) at the level of Lie algebras follows from the (CCP) theory developed for Hopf algebras and some extra technical constructions.

Introduction

Let $\mathcal{C}$ be a category whose objects are sets endowed with various algebraic, topological or differential structures. To illustrate, we can think of $\mathcal{C}$ as the category of groups, Lie groups, Lie algebras, algebras, Hopf algebras, locally compact quantum groups etc. Let E be an object of $\mathcal{C}$ and $A \subset E$ a given subobject of E. A subobject H of E is called a *complement* of A in E (or an A-*complement* of E) if E can be written as a 'product' of A and H such that A and H have 'minimal intersection' in E; the meaning of 'product' and 'minimal intersection' depends on the given category $\mathcal{C}$. We denote by $[E : A]^f$ the cardinal of the (possibly empty) isomorphism classes of all A-complements of E and we call it the *factorization index* of A in E. A natural problem which may be of interest for several areas of mathematics arises:

Classifying complements problem (CCP): *Let $A \subset E$ be two given objects in $\mathcal{C}$. If an A-complement of E exists, describe explicitly, classify all A-complements of E and compute the factorization index $[E : A]^f$.*

To start with, we consider the trivial case of the (CCP), namely when $\mathcal{C} = \mathcal{A}b$, the category of abelian groups. If A is a subgroup of E then an A-complement of E, if

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exists, is a subgroup $H \leq E$ such that $E = A + H$ and $A \cap H = \{0\}$. In this case the group H is of course unique up to an isomorphism. Thus, the factorization index $[E : A]^f \in \{0, 1\}$, for any $A \leq E \in \mathcal{A}b$. Now we take a step forward and formulate the (CCP) in the case when $\mathcal{C} = \mathcal{G}r$, the category of groups. In this context things change radically and the problem is far from being trivial. An A-complement in a group E is a subgroup $H \leq E$ such that $E = AH$ and $A \cap H = \{1\}$ - we use the multiplicative notation for arbitrary groups. In this case, we say that E *factorizes* through A and H and an A-complement of E is not necessarily unique. The basic example is the following: consider S_3 to be the symmetric group on three letters viewed as a subgroup in S_4 by considering 4 to be a fixed point. Then, the cyclic group C_4 and the Klein's group $C_2 \times C_2$ are both complements of S_3 in S_4 and thus the factorization index $[S_4 : S_3]^f = 2$. Hence, we expect to obtain a non-trivial and consistent theory of the (CCP) whose difficulty depends essentially on the category $\mathcal{C}$ as well as on the nature of the extension $A \subset E$ in the category $\mathcal{C}$. In the case when $\mathcal{C}$ is the category of groups the problem was recently solved in [4].

The aim of this paper is to give the full answer to the (CCP) if $\mathcal{C}$ is the category of Hopf algebras and then, as special cases, we obtain the answer to the same problem if $\mathcal{C}$ is the category of Lie algebras. We will see that even though the complements are not unique as in the case of abelian groups, it is enough to find only one complement H. All the other complements will be obtained from H by a certain type of deformation which we will introduce for the categories mentioned above. In order to explain our strategy of approaching the problem we will return to the group case. Let $A \leq E$ be a subgroup of E. The fact that a subgroup H of E is a complement of A in E is equivalent to the multiplication map $A \times H \to E$, $(a, h) \mapsto ah$ being bijective. In this case there exists a unique group structure on the cartesian product $A \times H$ such that the multiplication map becomes an isomorphism of groups and this group structure on $A \times H$ is precisely the Takeuchi's bicrossed product $A \bowtie H$ [21] associated to the canonical matched pair $(A, H, \triangleleft, \triangleright)$ of the factorization $E = AH$. Conversely, a group H is an A-complement of any bicrossed product of groups $A \bowtie H$. These simple observations raise to the level of principle valid for various categories $\mathcal{C}$: H is an A-complement of E in a given category $\mathcal{C}$ if and only if $E \cong A \bowtie H$, where $A \bowtie H$ is a '*bicrossed product*' in the category $\mathcal{C}$ associated to a '*matched pair*' between the objects A and H.[1] This principle becomes a theorem when $\mathcal{C}$ is the category of groups or groupoids [5], algebras [6], Hopf algebras [16], Lie groups or Lie algebras [14], locally compact quantum groups [22], multiplier Hopf algebras [7] and so on. Now, assume that H is a given A-complement of E. Hence, there exists a canonical isomorphism $A \bowtie H \cong E$ in $\mathcal{C}$ – as a general rule this isomorphism stabilizes A. Now, the description and the classification part of the (CCP) is obtained from the following subsequent question: describe and classify all objects $\mathbb{H}$ in $\mathcal{C}$ such that there exists an isomorphism $A \bowtie H \cong A \bowtie \mathbb{H}$ in $\mathcal{C}$ that stabilizes A. This is in fact a descent type problem, that can be called the *bicrossed descent theory* for the extension $A \subseteq E$ in the category $\mathcal{C}$. The classification of all A-complements of E needs a parallel theory which has to be developed, similar to what is called the *classification of forms*

[1] What a matched pair means and the construction of the bicrossed product in a given category $\mathcal{C}$ is part of the problem and needs to be treated separately for each category.

in the classical descent theory [13], [19]. Furthermore, the (CCP) is the converse of the *factorization problem* which asks for the description and classification of all objects E of $\mathcal{C}$ that factorize through two given objects A and H, i.e. of all possible bicrossed products $A \bowtie H$. For more details on the factorization problem we refer to [1].

The paper is organized as follows. In Section 1 we recall the basic concepts that will be used throughout the paper. In particular, we shall review the Majid's bicrossed product [15] associated to a matched pair of Hopf algebras $(A, H, \triangleleft, \triangleright)$. Section 2 offers the answer to the (CCP) problem for Hopf algebras. The answer will be given in four steps, each of them of interest in its own right, which we have called: the deformation of a Hopf algebra, the deformation of complements, the description of complements and the classification of complements. In Theorem 2.6 a general deformation of a given Hopf algebra H is introduced. This deformation H_r of H is associated to an arbitrary matched pair of Hopf algebras $(A, H, \triangleright, \triangleleft)$ and to a deformation map $r : H \to A$ in the sense of Definition 2.3. As a coalgebra $H_r = H$, with the new multiplication $\bullet$ defined by

$$h \bullet g := \big(h \triangleleft r(g_{(1)})\big)\, g_{(2)}$$

for all $h,\, g \in H_r = H$. Then H_r is a new Hopf algebra called the *r-deformation of H*. Theorem 2.8 proves that if $r : H \to A$ is a deformation map of a matched pair $(A, H, \triangleright, \triangleleft)$ of Hopf algebras, then H_r remains an A-complement of the bicrossed product $A \bowtie H$. Now, let $A \subseteq E$ be an extension of Hopf algebras and H a given A-complement of E. Then there exists a canonical matched pair of Hopf algebras $(A, H, \triangleleft, \triangleright)$ such that $E \cong A \bowtie H$. The description of all A-complements of E is given in Theorem 2.9: any other A-complement $\mathbb{H}$ of E is isomorphic as a Hopf algebra with an H_r, for some deformation map $r : H \to A$ of the canonical matched pair $(A, H, \triangleleft, \triangleright)$ associated to the A-complement H. This result proves that in order to find all complements of A in E it is enough to know only one A-complement: all the other A-complements are deformations of it. Finally, as a conclusion of previous results, Theorem 2.5 provides the classification of A-complements of E: there exists a bijection between the isomorphisms classes of all A-complements of E and a cohomological type object $\mathcal{HA}^2(H, A \,|\, (\triangleright, \triangleleft))$ and this bijection is explicitly described. In particular, we obtain the formula for computing the factorization index of a given extension $A \subseteq E$ of Hopf algebras: $[E : A]^f = |\mathcal{HA}^2(H, A \,|\, (\triangleright, \triangleleft))|$. In Section 3 we shall construct an example of a Hopf algebra extension of a given factorization index in Theorem 3.3. This is the extension $k[C_n] \subseteq H_{4n^2,\, \omega,\, \omega'}$, where $k[C_n]$ is the group algebra of a cyclic group of order n and $H_{4n^2,\, \omega,\, \omega'}$ is a $4n^2$-dimensional quantum group associated to two distinct n-th roots of unity ω and ω'.

In Section 4 we give the answer to the (CCP) in the case when $\mathcal{C}$ is the category of Lie algebras over a field k of characteristic zero. In this case we can derive some of the results as special cases of the theorems proved for Hopf algebras but, they do not follow in a straightforward manner. Although a result by Masuoka [17, Proposition 2.4] gives a bijective correspondence between the set of all matched pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ and the set of all matched pairs of Hopf algebras $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ there are still some obstacles to overcome. The first one comes from the fact that we no longer have a bijection between deformation maps for Hopf algebras and deformation maps for Lie algebras as in the group case. The second obstacle is to prove that any r-deformation

of an enveloping algebra $U(\mathfrak{g})$ in the sense of Theorem 2.6 is an enveloping algebra of a certain Lie algebra. It turns out that this Lie algebra is precisely the one introduced in Theorem 4.3 as a deformation of a given Lie algebra $\mathfrak{h}$: it is associated to a deformation map $r : \mathfrak{h} \to \mathfrak{g}$ of the matched pair $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$. This will allow us to use Theorem 2.9 in order to describe all complements of a Lie subalgebra in Theorem 4.7. Finally, the answer to the (CCP) for Lie algebras is proven in Theorem 4.9: if $\mathfrak{g}$ is a Lie subalgebra of Ξ and $\mathfrak{h}$ is a fixed $\mathfrak{g}$-complement of Ξ, then the isomorphism classes of all $\mathfrak{g}$-complements of Ξ are parameterized by a certain cohomological object denoted by $\mathcal{HA}^2(\mathfrak{h}, \mathfrak{g} \,|\, (\triangleright, \triangleleft))$.

1. Preliminaries

Unless explicitly specified otherwise, k will be an arbitrary field. All (co)algebras, Hopf algebras, Lie algebras, tensor products, homomorphisms and so on are over k. For a coalgebra C, we use Sweedler's Σ-notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc (summation understood). Let A and H be two Hopf algebras. H is called a right A-module coalgebra if H is a coalgebra in the monoidal category $\mathcal{M}_A$ of right A-modules, i.e. there exists $\triangleleft : H \otimes A \to H$ a morphism of coalgebras such that $(H, \triangleleft)$ is a right A-module. A morphism between two right A-module coalgebras $(H, \triangleleft)$ and $(H', \triangleleft')$ is a morphism of coalgebras $\psi : H \to H'$ that is also right A-linear. Furthermore, ψ is called unitary if $\psi(1_H) = 1_{H'}$. Similarly, A is a left H-module coalgebra if A is a coalgebra in the monoidal category of left H-modules, that is there exists $\triangleright : H \otimes A \to A$ a morphism of coalgebras such that $(A, \triangleright)$ is also a left H-module. The actions $\triangleleft : H \otimes A \to H$, $\triangleright : H \otimes A \to A$ are called *trivial* if $h \triangleleft a = \varepsilon_A(a)h$ and respectively $h \triangleright a = \varepsilon_H(h)a$, for all $a \in A$ and $h \in H$.

Let $A \subseteq E$ be an extension of Hopf subalgebras. A Hopf subalgebra $H \subseteq E$ is called a *complement* of A in E (or an A-*complement* of E) if the multiplication map $A \otimes H \to E$, $a \otimes h \mapsto a\,h$ is bijective. In this case we say that the Hopf algebra E factorizes through A and H. The bicrossed product of two Hopf algebras was introduced by Majid in [15, Proposition 3.12] under the name of double cross product in order to achieve a better understanding of the Drinfel'd double $D(H)$. We shall adopt the name of bicrossed product from [12, Theorem IX 2.3]. Let A and H be two Hopf algebras and $\triangleleft : H \otimes A \to H$, $\triangleright : H \otimes A \to A$ two morphisms of coalgebras such that the following normalizing conditions hold:

$$h \triangleright 1_A = \varepsilon_H(h)1_A, \quad 1_H \triangleright a = a, \quad 1_H \triangleleft a = \varepsilon_A(a)1_H, \quad h \triangleleft 1_A = h \tag{1}$$

for all $h \in H$, $a \in A$. We denote by $A \bowtie H$ the k-module $A \otimes H$ together with the multiplication:

$$(a \bowtie h) \cdot (c \bowtie g) := a(h_{(1)} \triangleright c_{(1)}) \bowtie (h_{(2)} \triangleleft c_{(2)})g \tag{2}$$

for all a, $c \in A$, h, $g \in H$, where we denoted $a \otimes h$ by $a \bowtie h$. The object $A \bowtie H$ is called the *bicrossed product of A and H* if $A \bowtie H$ is a Hopf algebra with the multiplication given by (2), the unit $1_A \bowtie 1_H$ and the coalgebra structure given by the tensor product of coalgebras. In this case, $(A, H, \triangleleft, \triangleright)$ is called a *matched pair* of Hopf algebras. The next theorem provides necessary and sufficient conditions for $A \bowtie H$ to be a bicrossed product.

Theorem 1.1. *Let A, H be two Hopf algebras and $\triangleleft : H \otimes A \to H$, $\triangleright : H \otimes A \to A$ two morphisms of coalgebras satisfying the normalizing conditions (1). The following statements are equivalent:*

(1) $A \bowtie H$ is a bicrossed product;

(2) $(H, \triangleleft)$ is a right A-module coalgebra, $(A, \triangleright)$ is a left H-module coalgebra and the following compatibilities hold for any a, $b \in A$, g, $h \in H$.

$$g \triangleright (ab) = (g_{(1)} \triangleright a_{(1)})\big((g_{(2)} \triangleleft a_{(2)}) \triangleright b\big) \tag{3}$$

$$(gh) \triangleleft a = \big(g \triangleleft (h_{(1)} \triangleright a_{(1)})\big)(h_{(2)} \triangleleft a_{(2)}) \tag{4}$$

$$g_{(1)} \triangleleft a_{(1)} \otimes g_{(2)} \triangleright a_{(2)} = g_{(2)} \triangleleft a_{(2)} \otimes g_{(1)} \triangleright a_{(1)} \tag{5}$$

In this case, $A \bowtie H$ has an antipode given by the formula:

$$S_{A \bowtie H}(a \bowtie h) = S_H(h_{(2)}) \triangleright S_A(a_{(2)}) \bowtie S_H(h_{(1)}) \triangleleft S_A(a_{(1)}) \tag{6}$$

for all $a \in A$ and $h \in H$.

Proof. (2) $\Rightarrow$ (1) This is just [16, Theorem 7.2.2] or [12, Theorem IX 2.3].

(1) $\Rightarrow$ (2) Follows as a special case of [2, Theorem 2.4] if we consider $f : H \otimes H \to A$ to be the trivial cocycle, i.e. $f(g, h) = \varepsilon_H(g)\varepsilon_H(h)$. See also [2, Examples 2.5] for details. □

From now on, in the light of Theorem 1.1, a matched pair of Hopf algebras will be viewed as a system $(A, H, \triangleleft, \triangleright)$, where $(H, \triangleleft)$ is a right A-module coalgebra, $(A, \triangleright)$ is a left H-module coalgebra such that the compatibility conditions (1) and (3)-(5) hold.

Examples 1.2. 1. Let $(A, \triangleright)$ be a left H-module coalgebra and consider H as a right A-module coalgebra via the trivial action, i.e. $h \triangleleft a = \varepsilon_A(a)h$. Then $(A, H, \triangleleft, \triangleright)$ is a matched pair of Hopf algebras if and only if $(A, \triangleright)$ is a left H-module algebra and the following compatibility condition holds

$$g_{(1)} \otimes g_{(2)} \triangleright a = g_{(2)} \otimes g_{(1)} \triangleright a \tag{7}$$

for all $g \in H$ and $a \in A$. In this case, the associated bicrossed product $A \bowtie H = A\#H$ is the semi-direct (smash) product of Hopf algebras as defined by Molnar [18] in the cocommutative case, for which the compatibility condition (7) holds automatically. Thus, $A\#H$ is the k-module $A \otimes H$, where the multiplication (2) takes the form:

$$(a\#h) \cdot (c\#g) := a\,(h_{(1)} \triangleright c)\,\#\,h_{(2)}g \tag{8}$$

for all a, $c \in A$, h, $g \in H$, where we denoted $a \otimes h$ by $a\#h$.

2. The fundamental example of a bicrossed product is the Drinfel'd double $D(H)$. Let H be a finite dimensional Hopf algebra. Then we have a matched pair of Hopf algebras $((H^*)^{\rm cop}, H, \triangleleft, \triangleright)$, where the actions $\triangleleft$ and $\triangleright$ are defined by:

$$h \triangleleft h^* := \langle h^*,\, S_H^{-1}(h_{(3)})h_{(1)}\rangle h_{(2)}, \quad h \triangleright h^* := \langle h^*,\, S_H^{-1}(h_{(2)})\, ?\, h_{(1)}\rangle \tag{9}$$

for all $h \in H$ and $h^* \in H^*$ ([12, Theorem IX.3.5]). The Drinfel'd double of H is the bicrossed product associated to this matched pair, i.e. $D(H) = (H^*)^{\rm cop} \bowtie H$. A generalization of the Drinfel'd double to the level of infinite dimensional Hopf algebras was introduced by Majid [16, Example 7.2.6] under the name of generalized quantum

double. This construction can be performed for any Hopf algebras A and H which are connected by a skew pairing $\lambda : H \otimes A \to k$.

3. Let G and K be two groups, $A = k[G]$ and $H = k[K]$ the corresponding group algebras. There exists a bijection between the set of all matched pairs of Hopf algebras $(k[G], k[K], \triangleleft, \triangleright)$ and the set of all matched pairs of groups $(G, K, \tilde{\triangleleft}, \tilde{\triangleright})$ in the sense of Takeuchi [21]. The bijection is given such that there exists an isomorphism of Hopf algebras $k[G] \bowtie k[K] \cong k[G \bowtie H]$, where $G \bowtie H$ is the Takeuchi's bicrossed product of groups ([12, pg. 207].

4. Let k be a field of characteristic zero, $\mathfrak{g}$ and $\mathfrak{h}$ two Lie algebras and $U(\mathfrak{g})$, $U(\mathfrak{h})$ the corresponding universal enveloping algebras. Then there is a bijective correspondence between the matched pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ as defined in [16, Definition 8.3.1] and the matched pairs of Hopf algebras $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$. Moreover, the Hopf algebra $U(\mathfrak{g}) \bowtie U(\mathfrak{h})$ obtained from the matched pair $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g} \bowtie \mathfrak{h})$ of the bicrossed product $\mathfrak{g} \bowtie \mathfrak{h}$ of Lie algebras associated to the matched pair $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ ([17, Proposition 2.4]).

A bicrossed product $A \bowtie H$ will be viewed as a left A-module via the restriction of scalars through the canonical inclusion $i_A : A \to A \bowtie H$, $i_A(a) = a \bowtie 1_H$, for all $a \in A$. The next result is due to Majid [16, Theorem 7.2.3].

Theorem 1.3. *Let A, H be two Hopf algebras. A Hopf algebra E factorizes through A and H if and only if there exists a matched pair of Hopf algebras $(A, H, \triangleleft, \triangleright)$ such that the multiplication map*

$$m_E : A \bowtie H \to E, \qquad m_E(a \bowtie h) = ah$$

for all $a \in A$ and $h \in H$ is an isomorphism of Hopf algebras. Moreover, if E factorizes through A and H then the actions of the matched pair $(A, H, \triangleleft, \triangleright)$ are constructed as follows for all $a \in A$, $h \in H$:

$$ha = (h_{(1)} \triangleright a_{(1)})(h_{(2)} \triangleleft a_{(2)}) \tag{10}$$

Furthermore, in this case the isomorphism of Hopf algebras $m_E : A \bowtie H \to E$ is also a left A-module map.

From now on, the matched pair constructed in (10) will be called the *canonical matched pair* associated to the factorization of E through A and H. We use this notation in order to distinguish this matched pair among other possible matched pairs $(A, H, \triangleright', \triangleleft')$ such that $A \bowtie' H \cong E$, where $A \bowtie' H$ is the bicrossed product associated to the matched pair $(A, H, \triangleright', \triangleleft')$. The following is just the formal dual of the notion of central map:

Definition 1.4. Let A and H be two Hopf algebras. A coalgebra map $r : H \to A$ is called *cocentral* if the following compatibility holds:

$$r(h_{(1)}) \otimes h_{(2)} = r(h_{(2)}) \otimes h_{(1)} \tag{11}$$

for all $h \in H$.

The set $CoZ(H, A)$ of all cocentral maps is a group with respect to the convolution product. We denote by $CoZ^1(H, A)$ the subgroup of $CoZ(H, A)$ of all cocentral maps $r : H \to A$ such that $r(1_H) = 1_A$.

2. Classifying complements for Hopf algebras

Let $A \subseteq E$ be a given extension of Hopf subalgebras. In what follows we denote by $\mathcal{F}(A, E)$ the (possibly empty) isomorphism classes of all A-complements of E. The problem of existence of A-complements of E has to be treated 'case by case' for every given Hopf algebra extension $A \subseteq E$, a computational part of it can not be avoided. This was also the approach used in the similar problem at the level of groups, i.e. corresponding to the Hopf algebra extension $k[A] \subseteq k[G]$, for two groups A and G with $A \leq G$ (see [11] and the references therein). For example, if $E = k[A_6]$ and A is a proper Hopf subalgebra, then $\mathcal{F}(A, k[A_6])$ is the empty set. This is based on the fact that the alternating group A_6 has no proper factorizations [23].

In this section we shall give the complete answer to the (CCP) for Hopf algebras, i.e. we shall describe and classify of all A-complements of E. First of all, we remark that if $(A, H, \triangleright, \triangleleft)$ is a matched pair of Hopf algebras, then $H \cong \{1_A\} \bowtie H$ is an A-complement of $A \bowtie H$, where we consider $A \cong A \bowtie \{1_H\} \subseteq A \bowtie H$. Now, let H be a given A-complement of E and $(A, H, \triangleleft, \triangleright)$ the associated canonical matched pair as in Theorem 1.3; thus the multiplication map $A \bowtie H \to E$, $a \bowtie h \mapsto ah$ is a Hopf algebra isomorphism that is also a left A-module map. This elementary observation will be important below. We shall describe all A-complements H' of E in terms of $(H, \triangleleft, \triangleright)$ and a unitary cocentral map $r : H \to A$ satisfying a certain compatibility condition. The classification of all A-complements of E is also given by proving that the set $\mathcal{F}(A, E)$ is in bijection to a cohomological object that will be denoted by $\mathcal{HA}^2(H, A \,|\, (\triangleright, \triangleleft))$. In order to prove these results we need to introduce a few more concepts:

Definition 2.1. Let A be a Hopf subalgebra of E. We define the *factorization index* of A in E as the cardinal of $\mathcal{F}(A, E)$ and it will be denoted by $[E : A]^f = |\, \mathcal{F}(A, E)\,|$. The extension $A \subseteq E$ is called *rigid* if $[E : A]^f = 1$. We shall write $[E : A]^f = 0$, if $\mathcal{F}(A, E)$ is empty.

We write down explicitly what a rigid extension of Hopf algebras E/A means: $[E : A]^f = 1$ if and only if any two A-complements H and H' of E are isomorphic as Hopf algebras. Equivalently, this can be restated as follows: if $E \cong A \bowtie H \cong A \bowtie' H'$ (isomorphism of Hopf algebras and left A-modules), then $H \cong H'$. This is a Krull-Schmidt-Azumaya type theorem for bicrossed products of Hopf algebras.

Examples 2.2. 1. In most of the cases, for a given extension of Hopf algebras $A \subseteq E$ the factorization index $[E : A]^f$ is equal to 0 (i.e. there exists no A-complement of E) or 1. For instance, above we have shown in fact that $[k[A_6] : A]^f = 0$, for any proper Hopf subalgebra A of the group Hopf algebra $k[A_6]$.

2. A general example of a rigid extension of Hopf algebra will be given in Corollary 2.10: if $A\#H$ is an arbitrary semidirect product of two Hopf algebras, then the extension $A \subset A\#H$ is rigid. A detailed example is presented in Corollary 3.1.

3. Examples of extensions E/A for which $[E : A]^f \geq 2$ are quite rare, which makes them tempting to identify. For instance, the extension $k[S_3] \subseteq k[S_4]$ has factorization index 2 as we pointed out in the introduction. We shall provide an elaborated way of

constructing examples of Hopf algebra extensions E/A of a given factorization index in Theorem 3.2.

Definition 2.3. Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. A unitary cocentral map $r \in CoZ^1(H, A)$ is called a *deformation map* of the matched pair $(A, H, \triangleright, \triangleleft)$ if the following compatibility holds:

$$r\Big(\big(h \triangleleft r(g_{(1)})\big)\, g_{(2)}\Big) = r(h_{(1)})\,\big(h_{(2)} \triangleright r(g)\big) \tag{12}$$

for all $g,\, h \in H$.

Let $\mathcal{DM}\,(H, A\,|\,(\triangleright, \triangleleft)) \subseteq CoZ^1(H, A)$ be the set of all deformation maps of the matched pair $(A, H, \triangleright, \triangleleft)$. The trivial map $r : H \to A$, $r(h) = \varepsilon(h)1_A$ is a deformation map. We shall introduce now the following:

Definition 2.4. Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras. Two deformation maps $r,\, R : H \to A$ are called *equivalent* and we denote this by $r \sim R$ if there exists $\sigma : H \to H$ an unitary automorphism of the coalgebra H such that

$$\sigma\big((h \triangleleft r(g_{(1)}))\, g_{(2)}\big) = \big(\sigma(h) \triangleleft R(\sigma(g_{(1)}))\big)\, \sigma(g_{(2)}) \tag{13}$$

for all $g,\, h \in H$.

The theorem that gives the answer to the (CCP) for Hopf algebras is the following:

Theorem 2.5. ***(Classification of complements)*** *Let A be a Hopf subalgebra of E, H an A-complement of E and $(A, H, \triangleright, \triangleleft)$ the associated canonical matched pair. Then:*

(1) $\sim$ *is an equivalence relation on $\mathcal{DM}\,(H, A\,|\,(\triangleright, \triangleleft))$. We denote by $\mathcal{HA}^2(H, A\,|\,(\triangleright, \triangleleft))$ the quotient set $\mathcal{DM}\,(H, A\,|\,(\triangleright, \triangleleft))/\sim$.*

(2) *There exists a bijection between the isomorphism classes of all A-complements of E and $\mathcal{HA}^2(H, A\,|\,(\triangleright, \triangleleft))$. In particular, the factorization index of A in E is computed by the formula:*

$$[E : A]^f = |\mathcal{HA}^2(H, A\,|\,(\triangleright, \triangleleft))|$$

This result shows the following: in order to describe and to classify the set of all A-complements of E it is enough to know only one object in $H \in \mathcal{F}(A, E)$. We prove this theorem in three steps which we have called: the deformation of a Hopf algebra, the deformation of complements and finally the description of complements. First we prove the following theorem where a general deformation of a given Hopf algebra H is proposed. This deformation is associated to an arbitrary matched pair of Hopf algebras $(A, H, \triangleright, \triangleleft)$ and to a deformation map $r : H \to A$.

Theorem 2.6. ***(Deformation of a Hopf algebra)*** *Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras and $r : H \to A$ a deformation map. Let $H_r := H$, as a coalgebra, with the new multiplication $\bullet$ on H defined for any $h,\, g \in H$ as follows:*

$$h \bullet g := \big(h \triangleleft r(g_{(1)})\big)\, g_{(2)} \tag{14}$$

Then $H_r = (H_r, \bullet, 1_H, \Delta_H, \varepsilon_H)$ is a Hopf algebra with the antipode given by

$$S : H_r \to H_r, \qquad S(h) := S_H(h_{(2)}) \triangleleft (S_A \circ r)(h_{(1)}) \tag{15}$$

for all $h \in H$, called the r-deformation of H.

Proof. Using the normalizing conditions (1) and the fact that $r : H \to A$ is a unitary map, 1_H remains the unit for the new multiplication $\bullet$ given by (14). On the other hand for any $h, g, t \in H$ we have:

$$\begin{aligned}
(h \bullet g) \bullet t &= \big[\big(h \triangleleft r(g_{(1)})\big) g_{(2)}\big] \bullet t\\
&= \Big(\underline{\big((h \triangleleft r(g_{(1)}))g_{(2)}\big) \triangleleft r(t_{(1)})}\Big)\, t_{(2)}\\
&\overset{(4)}{=} \Big(\underline{\big(h \triangleleft r(g_{(1)})\big) \triangleleft \big(g_{(2)} \triangleright r(t_{(1)})\big)}\Big)\big(g_{(3)} \triangleleft r(t_{(2)})\big)\, t_{(3)}\\
&= \Big(h \triangleleft \big(\underline{r(g_{(1)})(g_{(2)} \triangleright r(t_{(1)}))}\big)\Big)\big(g_{(3)} \triangleleft r(t_{(2)})\big)\, t_{(3)}\\
&\overset{(12)}{=} \Big(h \triangleleft r\big((g_{(1)} \triangleleft r(t_{(1)}))\, \underline{t_{(2)}}\big)\Big)\, \big(g_{(2)} \triangleleft \underline{r(t_{(3)})}\big)\, t_{(4)}\\
&\overset{(11)}{=} \Big(h \triangleleft r\big((g_{(1)} \triangleleft r(t_{(1)}))\, t_{(3)}\big)\Big)\big(g_{(2)} \triangleleft r(t_{(2)})\big)\, t_{(4)}\\
&= h \bullet \big[\big(g \triangleleft r(t_{(1)})\big)\, t_{(2)}\big]\\
&= h \bullet (g \bullet t)
\end{aligned}$$

Thus, the multiplication $\bullet$ is associative and has 1_H as a unit. Next we prove that $\varepsilon_H : H \to k$ and $\Delta_H : H \to H \otimes H$ are algebra maps with respect to the new multiplication (14). Indeed, for any $h, g \in H$ we have:

$$\varepsilon_H(h \bullet g) = \varepsilon_H\big[\big(h \triangleleft r(g_{(1)})\big)\, g_{(2)}\big] = \varepsilon_H(h)\varepsilon_H(g_{(1)})\varepsilon_H(g_{(2)}) = \varepsilon_H(h)\varepsilon_H(g)$$

and

$$\begin{aligned}
\Delta_H(h \bullet g) &= \Delta_H\Big(\big(h \triangleleft r(g_{(1)})\big)\, g_{(2)}\Big)\\
&= \big(h_{(1)} \triangleleft r(g_{(1)(1)})\big)\, g_{(2)(1)} \otimes \big(h_{(2)} \triangleleft r(g_{(1)(2)})\big)\, g_{(2)(2)}\\
&= \big(h_{(1)} \triangleleft r(g_{(1)})\big)\, \underline{g_{(3)}} \otimes \big(h_{(2)} \triangleleft \underline{r(g_{(2)})}\big)\, g_{(4)}\\
&\overset{(11)}{=} \big(h_{(1)} \triangleleft r(g_{(1)})\big)\, g_{(2)} \otimes \big(h_{(2)} \triangleleft r(g_{(3)})\big)\, g_{(4)}\\
&= h_{(1)} \bullet g_{(1)} \otimes h_{(2)} \bullet g_{(2)}
\end{aligned}$$

Thus $H_r = (H_r, \bullet, 1_H, \Delta_H, \varepsilon_H)$ is a bialgebra. It remains to prove that $S : H_r \to H_r$ given by (15) is an antipode for H_r. Indeed, for any $h \in H$ we have:

$$\begin{aligned}
S(h_{(1)}) \bullet h_{(2)} &= \big(S_H(h_{(2)}) \triangleleft S_A \circ r(h_{(1)})\big) \bullet h_{(3)}\\
&= \Big(\big(S_H(\underline{h_{(2)}}) \triangleleft S_A \circ r(h_{(1)})\big) \triangleleft \underline{r(h_{(3)})}\Big)\, h_{(4)}\\
&\overset{(11)}{=} \Big(\big(S_H(h_{(3)}) \triangleleft S_A \circ r(h_{(1)})\big) \triangleleft r(h_{(2)})\Big)\, h_{(4)}\\
&= \Big(S_H(h_{(3)}) \triangleleft \big(S_A \circ r(h_{(1)}) r(h_{(2)})\big)\Big)\, h_{(4)}\\
&= S_H(h_{(1)})\, h_{(2)}\\
&= \varepsilon_H(h) 1_H
\end{aligned}$$

for all $h \in H$, where we use the fact that $\big(S_A \circ r(h_{(1)})\big) r(h_{(2)}) = \varepsilon_H(h) 1_A$. Analogous, one can show that $h_{(1)} \bullet S(h_{(2)}) = \varepsilon_H(h) 1_H$, for all $h \in H$ and the proof is finished. $\square$

Remarks 2.7. 1. Assume that in Theorem 2.6 the unitary cocentral map $r : H \to A$ is the trivial one $r(h) = \varepsilon_H(h) 1_A$ or the right action $\triangleleft : H \otimes A \to H$ of A on H is the trivial action, i.e. $h \triangleleft a = \varepsilon_A(a) h$, for all $h \in H$ and $a \in A$. Then $H_r = H$ as Hopf algebras. In general, the new Hopf algebra H_r may not be isomorphic to H as a Hopf algebra. Indeed, $k[C_2 \times C_2]$ and $k[C_4]$ are both $k[S_3]$-complements of $k[S_4]$. Using Theorem 2.9 proven below, we obtain that the Hopf algebra $k[C_2 \times C_2]$ is isomorphic to $k[C_4]_r$, for some deformation map $r : k[C_4] \to k[S_3]$. A more detailed example of Hopf algebras H_r which are not isomorphic to H will be provided in Theorem 3.3.

2. We should notice that there are two other famous deformations of a given Hopf algebra in the literature. The first one was introduced by Drinfel'd [10]: the comultiplication of a Hopf algebra H is deformed using an invertible element $R \in H \otimes H$, called twist, in order to obtain a new Hopf algebra H^R. The dual case was introduced by Doi [8]: the algebra structure of a Hopf algebra H was deformed using a Sweedler cocycle $\sigma : H \otimes H \to k$ as follows: let $H_\sigma = H$, as a coalgebra, with the new multiplication given by

$$h \cdot g := \sigma(h_{(1)}, g_{(1)})\, h_{(2)} g_{(2)}\, \sigma^{-1}(h_{(3)}, g_{(3)})$$

for all h, $g \in H$. Then H_σ is a new Hopf algebra [8, Theorem 1.6] and among several interesting applications for quantum groups we mention that the Drinfel'd double $D(H)$ is a special case of this deformation [9].

The next step shows that if H is a given A-complement of E, then H_r remains an A-complement of E, for any deformation map $r : H \to A$.

Theorem 2.8. ***(Deformation of complements)*** *Let $(A, H, \triangleright, \triangleleft)$ be a matched pair of Hopf algebras, $r : H \to A$ a deformation map and H_r the r-deformation of H. We define a new action $\triangleright^r : H_r \otimes A \to A$ given as follows for any $h \in H_r$ and $a \in A$:*

$$h \triangleright^r a \;\; := \;\; r(h_{(1)}) \big(h_{(2)} \triangleright a_{(1)}\big) (S_A \circ r)(h_{(3)} \triangleleft a_{(2)}) \tag{16}$$

Then $(A, H_r, \triangleright^r, \triangleleft)$ is a matched pair of Hopf algebras and the k-linear map

$$\psi : A \bowtie^r H_r \to A \bowtie H, \quad \psi(a \bowtie^r h) = a\, r(h_{(1)}) \bowtie h_{(2)} \tag{17}$$

for all $a \in A$ and $h \in H$ is a left A-linear Hopf algebra isomorphism, where $A \bowtie^r H_r$ is the bicrossed product associated to the matched pair $(A, H_r, \triangleright^r, \triangleleft)$.

In particular, H_r is an A-complement of $A \bowtie H$.

Proof. Instead of using a very long computation to prove that $(A, H_r, \triangleright^r, \triangleleft)$ is a matched pair of Hopf algebras we use Theorem 1.1. In order to do this we first prove that $\triangleright^r : H_r \otimes A \to A$ given by (16) is a coalgebra map. Indeed, for any $a \in A$, $h \in H$ we

have:

$$\begin{aligned}
(h \triangleright^r a)_{(1)} \otimes (h \triangleright^r a)_{(2)} &= r(h_{(1)})\big(\underline{h_{(3)}} \triangleright a_{(1)}\big)\big(S_A \circ r\big)\big(h_{(6)} \triangleleft a_{(4)}\big)\otimes \\
&\quad \underline{r(h_{(2)})}\big(h_{(4)} \triangleright a_{(2)}\big)\big(S_A \circ r\big)\big(h_{(5)} \triangleleft a_{(3)}\big) \\
&\overset{(11)}{=} r(h_{(1)})\big(h_{(2)} \triangleright a_{(1)}\big)\big(S_A \circ \underline{r)\big(h_{(6)} \triangleleft a_{(4)}}\big)\otimes \\
&\quad r(h_{(3)})\big(h_{(4)} \triangleright a_{(2)}\big)\big(S_A \circ r\big(\underline{h_{(5)} \triangleleft a_{(3)}}\big) \\
&\overset{(11)}{=} r(h_{(1)})\big(h_{(2)} \triangleright a_{(1)}\big)\big(S_A \circ r\big)\big(\underline{h_{(5)} \triangleleft a_{(3)}}\big)\otimes \\
&\quad r(h_{(3)})\big(\underline{h_{(4)} \triangleright a_{(2)}}\big)\big(S_A \circ r\big)\big(h_{(6)} \triangleleft a_{(4)}\big) \\
&\overset{(5)}{=} r(h_{(1)})\big(h_{(2)} \triangleright a_{(1)}\big)\big(S_A \circ r\big)\big(\underline{h_{(4)}} \triangleleft a_{(2)}\big)\otimes \\
&\quad \underline{r(h_{(3)})}\big(h_{(5)} \triangleright a_{(3)}\big)\big(S_A \circ r\big)\big(h_{(6)} \triangleleft a_{(4)}\big) \\
&\overset{(11)}{=} r(h_{(1)})\big(h_{(2)} \triangleright a_{(1)}\big)\big(S_A \circ r\big)\big(h_{(3)} \triangleleft a_{(2)}\big)\otimes \\
&\quad r(h_{(4)})\big(h_{(5)} \triangleright a_{(3)}\big)\big(S_A \circ r\big)\big(h_{(6)} \triangleleft a_{(4)}\big) \\
&= h_{(1)} \triangleright^r a_{(1)} \otimes h_{(2)} \triangleright^r a_{(2)}
\end{aligned}$$

Thus, $\triangleright^r : H_r \otimes A \to A$ is a coalgebra map and moreover, the normalizing conditions (1) are trivially fulfilled for the pair of actions $(\triangleright^r, \triangleleft)$. In order to prove that $(A, H_r, \triangleright^r, \triangleleft)$ is a matched pair of Hopf algebras we use Theorem 1.1 as follows: first, observe that the map $\psi : A \otimes H_r \to A \bowtie H$, $\psi(a \otimes h) = a\, r(h_{(1)}) \bowtie h_{(2)}$ is an unitary isomorphism of coalgebras with the inverse given by:

$$\psi^{-1} : A \bowtie H \to A \otimes H_r, \quad \psi^{-1}(a \bowtie h) = a\,(S_A \circ r)(h_{(1)}) \otimes h_{(2)}$$

for all $a \in A$ and $h \in H$. There exists a unique algebra structure $\diamond$ on the coalgebra $A \otimes H_r$ such that ψ becomes an isomorphism of Hopf algebras and this is obtained by transferring the algebra structure from the Hopf algebra $A \bowtie H$ via the isomorphism of coalgebras ψ, i.e. is given by:

$$(a \otimes h) \diamond (b \otimes g) := \psi^{-1}\big(\psi(a \otimes h)\, \psi(b \otimes g)\big)$$

for all a, $b \in A$ and h, $g \in H_r = H$. If we prove that this algebra structure $\diamond$ on the tensor product of coalgebras $A \otimes H_r$ is exactly the one given by (2) associated to the new pair of actions $(\triangleright^r, \triangleleft)$ on a bicrossed product then the proof is finished by using Theorem 1.1. Indeed, for any a, $b \in A$ and g, $h \in H$ we have:

$$\begin{aligned}
& (a \otimes h) \diamond (b \otimes g) = \psi^{-1}\big(\psi(a \otimes h)\, \psi(b \otimes g)\big) \\
= \;& \psi^{-1}\Big(\big(a\, r(h_{(1)}) \bowtie h_{(2)}\big)\big(b\, r(g_{(1)}) \bowtie g_{(2)}\big)\Big) \\
= \;& \psi^{-1}\Big(a\, r(h_{(1)})\big(h_{(2)} \triangleright b_{(1)} r(g_{(1)})\big) \bowtie \big(h_{(3)} \triangleleft b_{(2)} r(g_{(2)})\big)\, g_{(3)}\Big) \\
= \;& a\, r(h_{(1)})\big(h_{(2)} \triangleright b_{(1)} r(g_{(1)})\big)(S_A \circ r)\Big(\big(h_{(3)} \triangleleft b_{(2)} r(g_{(2)})\big)\, \underline{g_{(4)}}\Big) \otimes \\
& \big(h_{(4)} \triangleleft b_{(3)} \underline{r(g_{(3)})}\big)\, g_{(5)}
\end{aligned}$$

$$
\begin{aligned}
&\overset{(11)}{=} && a\,r(h_{(1)})\big(h_{(2)} \triangleright b_{(1)}r(g_{(1)})\big)(S_A \circ r)\Big(\big(h_{(3)} \triangleleft b_{(2)}r(g_{(2)})\big)\, g_{(3)}\Big) \otimes \\
& && \big(h_{(4)} \triangleleft b_{(3)}r(g_{(4)})\big)\, g_{(5)} \\
&= && a\,r(h_{(1)})\big(h_{(2)} \triangleright b_{(1)}r(g_{(1)})\big)(S_A \circ \underline{r)\Big(\big((h_{(3)} \triangleleft b_{(2)}) \triangleleft r(g_{(2)})\big)\, g_{(3)}}\Big) \otimes \\
& && \big(h_{(4)} \triangleleft b_{(3)}r(g_{(4)})\big)\, g_{(5)} \\
&\overset{(12)}{=} && a\,r(h_{(1)})\big(\underline{h_{(2)} \triangleright b_{(1)}r(g_{(1)})}\big)S_A\Big[r\big(h_{(3)} \triangleleft b_{(2)}\big)\Big(\big(h_{(4)} \triangleleft b_{(3)}\big)\triangleright r(g_{(2)})\Big)\Big] \otimes \\
& && \big(h_{(5)} \triangleleft b_{(4)}r(g_{(3)})\big)\, g_{(4)} \\
&\overset{(3)}{=} && a\,r(h_{(1)})\big(h_{(2)} \triangleright b_{(1)}\big)\Big(\big(h_{(3)} \triangleleft b_{(2)}\big)\triangleright r(g_{(1)})\Big)S_A\Big(\big(\underline{h_{(5)} \triangleleft b_{(4)}}\big)\triangleright r(g_{(2)})\Big) \\
& && (S_A \circ \underline{r\big(h_{(4)} \triangleleft b_{(3)}\big)} \otimes \big(h_{(6)} \triangleleft b_{(5)}r(g_{(3)})\big)\, g_{(4)} \\
&\overset{(11)}{=} && a\,r(h_{(1)})\big(h_{(2)} \triangleright b_{(1)}\big)\underline{\Big(\big(h_{(3)} \triangleleft b_{(2)}\big)\triangleright r(g_{(1)})\Big)S_A\Big(\big(h_{(4)} \triangleleft b_{(3)}\big)\triangleright r(g_{(2)})\Big)} \\
& && (S_A \circ r)\big(h_{(5)} \triangleleft b_{(4)}\big) \otimes \big(h_{(6)} \triangleleft b_{(5)}r(g_{(3)})\big)\, g_{(4)} \\
&= && a\,r(h_{(1)})\big(h_{(2)} \triangleright b_{(1)}\big)(S_A \circ r)\big(h_{(3)} \triangleleft b_{(2)}\big) \otimes \big(h_{(4)} \triangleleft b_{(3)}r(g_{(1)})\big)\, g_{(2)} \\
&= && a\,r(h_{(1)})\big(h_{(2)} \triangleright b_{(1)}\big)(S_A \circ r)\big(h_{(3)} \triangleleft b_{(2)}\big) \otimes \Big(h_{(4)} \triangleleft b_{(3)}\Big)\bullet g \\
&= && a\,(h_{(1)} \triangleright^r b_{(1)}) \otimes (h_{(2)} \triangleleft b_{(2)}) \bullet g = (a \bowtie^r h)(b \bowtie^r g)
\end{aligned}
$$

where $\bullet$ is the multiplication on H_r. The proof is now completely finished. □

Now, we shall prove the converse of Theorem 2.8: if H is a given A-complement of E then any other A-complement $\mathbb{H}$ is isomorphic with some H_r, for some deformation map $r : H \to A$.

Theorem 2.9. ***(Description of complements)*** *Let A be a Hopf subalgebra of E, H an A-complement of E with the associated canonical matched pair $(A, H, \triangleright, \triangleleft)$ and let $\mathbb{H}$ be an arbitrary A-complement of E. Then there exists an isomorphism of Hopf algebras $\mathbb{H} \cong H_r$, for some deformation map $r : H \to A$ of the matched pair $(A, H, \triangleright, \triangleleft)$.*

Proof. It follows from Theorem 1.3 that the multiplication map $m_E : A \bowtie H \to E$ is a left A-linear Hopf algebra isomorphism. Let $(A, \mathbb{H}, \triangleright', \triangleleft')$ be the canonical matched pair associated to $\mathbb{H}$; thus the multiplication map $m'_E : A \bowtie' \mathbb{H} \to E$ is a a left A-linear Hopf algebra isomorphism. Then $\psi := m_E^{-1} \circ m'_E : A \bowtie' \mathbb{H} \to A \bowtie H$ is a left A-linear Hopf algebra isomorphism as a composition of such maps. Now we apply [1, Theorem 3.5]: this left A-linear Hopf algebra isomorphism $\psi : A \bowtie' \mathbb{H} \to A \bowtie H$ is uniquely determined by a pair $(\overline{r}, \overline{v})$ consisting of a unitary cocentral map $\overline{r} : \mathbb{H} \to A$ and a unitary isomorphism

of coalgebras $\overline{v} : \mathbb{H} \to H$ satisfying the following compatibility conditions:

$$\overline{r}(h'g') = \overline{r}(h'_{(1)}) \left(\overline{v}(h'_{(2)}) \triangleright \overline{r}(g')\right) \tag{18}$$

$$\overline{v}(h'g') = \left(\overline{v}(h') \triangleleft \overline{r}(g'_{(1)})\right) \overline{v}(g'_{(2)}) \tag{19}$$

$$h' \triangleright' a = \overline{r}(h'_{(1)}) \left(\overline{v}(h'_{(2)}) \triangleright a_{(1)}\right) (S_A \circ \overline{r})(h'_{(3)} \triangleleft' a_{(2)}) \tag{20}$$

$$\overline{v}(h' \triangleleft' a) = \overline{v}(h') \triangleleft a \tag{21}$$

for all h', $g' \in \mathbb{H}$ and $a \in A$. Moreover, $\psi : A \bowtie' \mathbb{H} \to A \bowtie H$ is given by:

$$\psi(a \bowtie' h') = a\, \overline{r}(h'_{(1)}) \bowtie \overline{v}(h'_{(2)})$$

for all $a \in A$ and $h' \in \mathbb{H}$. We define now

$$r : H \to A, \quad r := \overline{r} \circ \overline{v}^{-1}$$

If we prove that r is a deformation map of the matched pair $(A, H, \triangleright, \triangleleft)$ and $\overline{v} : \mathbb{H} \to H_r$ is a Hopf algebra isomorphism the proof is finished. First, notice that r is a unitary cocentral map as $\overline{r}$, $\overline{v}$ are both unitary coalgebra maps and $\overline{r}$ is a unitary cocentral map. We have to show that the compatibility condition (12) holds for r. Indeed, from (18) and (19) we obtain:

$$\overline{r} \circ \overline{v}^{-1}\left[\left(\overline{v}(h') \triangleleft \overline{r}(g'_{(1)})\right) \overline{v}(g'_{(2)}) \right] = \overline{r}(h'_{(1)}) \left(\overline{v}(h'_{(2)}) \triangleright \overline{r}(g')\right) \tag{22}$$

for all h', $g' \in \mathbb{H}$. Now let h, $g \in H$ and write the compatibility condition (22) for $h' = \overline{v}^{-1}(h)$ and $g' = \overline{v}^{-1}(g)$. We obtain

$$r\Big(\left(h \triangleleft r(g_{(1)})\right) g_{(2)}\Big) = r(h_{(1)}) \left(h_{(2)} \triangleright r(g)\right)$$

that is (12) holds and hence $r : H \to A$ is a deformation map. Finally, $\overline{v} : \mathbb{H} \to H_r$ is a coalgebra isomorphism as the coalgebra structure on H_r coincides with the one on H. Hence, we are left to prove that $\overline{v}$ is also an algebra map. Indeed, for any h', $g' \in \mathbb{H}$ we have:

$$\overline{v}(h'g') \overset{(19)}{=} \left(\overline{v}(h') \triangleleft \overline{r}(g'_{(1)})\right) \overline{v}(g'_{(2)}) \overset{(14)}{=} \overline{v}(h') \bullet \overline{v}(g')$$

where we denoted by $\bullet$ the multiplication on H_r as defined by (14). Hence $\overline{v} : \mathbb{H} \to H_r$ is a Hopf algebra isomorphism and the proof is finished. □

In the next corollary $A\#H$ is the semidirect (smash) product of two Hopf algebras in the sense of Example 1.2 associated to a given action $\triangleright : H \otimes A \to A$.

Corollary 2.10. *Let $A\#H$ be an arbitrary semidirect product of two Hopf algebras. Then the factorization index $[A\#H : A]^f = 1$, i.e. the extension $A \to A\#H$ is rigid.*

Proof. Indeed, $H \cong \{1_A\}\#H$ is an A-complement of the semidirect product $A\#H$. Moreover, the right action $\triangleleft$ of the canonical matched pair $(A, H, \triangleright, \triangleleft)$ constructed in (10) for the factorization of $A\#H$ through $A \cong A\#\{1_H\}$ and $H \cong \{1_A\}\#H$ is the trivial action. Thus, using Remark 2.7, any r-deformation of $H \cong \{1_A\}\#H$ coincides with H. Now the conclusion follows by using Theorem 2.9. □

We are now ready to prove Theorem 2.5:

The proof of Theorem 2.5. It follows from Theorem 2.9 that if $\mathbb{H}$ is an arbitrary A-complement of E, then there exists an isomorphism of Hopf algebras $\mathbb{H} \cong H_r$, for some deformation map $r : H \to A$. Thus, in order to classify all A-complements of E we can consider only r-deformations of H, for various deformation maps $r : H \to A$.

Now let r, $R : H \to A$ be two deformation maps. As the coalgebra structure on H_r and H_R coincide with the one of H, we obtain that the Hopf algebras H_r and H_R are isomorphic if and only if there exists $\sigma : H \to H$ a unitary coalgebra isomorphism such that $\sigma : H_r \to H_R$ is also an algebra map. Taking into account the definition of the multiplication on H_r given by (14) we obtain that σ is an algebra map if and only if the compatibility condition (13) of Definition 2.4 holds, i.e. $r \sim R$. Hence, $r \sim R$ if and only if $\sigma : H_r \to H_r$ is an isomorphism of Hopf algebras. Thus we obtain that $\sim$ is an equivalence relation on $\mathcal{DM}(H, A \,|\, (\triangleright, \triangleleft))$ and the map

$$\mathcal{HA}^2(H, A \,|\, (\triangleright, \triangleleft)) \to \mathcal{F}(A, E), \qquad \overline{r} \mapsto H_r$$

where $\overline{r}$ is the equivalence class of r via the relation $\sim$, is well defined and a bijection between sets. This finishes the proof. □

3. Examples

In this section we shall provide an example of a Hopf algebra extension $A \subseteq E$ whose factorization index is arbitrary large.

For a positive integer n we denote by $U_n(k) = \{\omega \in k \,|\, \omega^n = 1\}$ the cyclic group of n-th roots of unity in k and by $\nu(n) = |U_n(k)|$ the order of the group $U_n(k)$. If $\nu(n) = n$, then any generator of $U_n(k)$ is called a primitive n-th root of unity. From now on, C_n will denote the cyclic group of order n generated by c or d (if we consider two copies of C_n) and k will be a field of characteristic $\neq 2$. Let $A := H_4$ be the Sweedler's 4-dimensional Hopf algebra generated by g and x subject to the relations:

$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx$$

with the coalgebra structure given such that g is a group-like element and x is $(1, g)$-primitive, that is

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x$$

It was proven in [1, Proposition 4.3] that there exists a bijective correspondence between the set of all matched pairs $(H_4, k[C_n], \triangleleft, \triangleright)$ and the group $U_n(k)$ such that the matched pair $(H_4, k[C_n], \triangleleft, \triangleright)$ associated to $\omega \in U_n(k)$ is given as follows: $\triangleleft : k[C_n] \otimes H_4 \to k[C_n]$ is the trivial action and $\triangleright : k[C_n] \otimes H_4 \to H_4$ is defined by:

$$c^i \triangleright g = g, \quad c^i \triangleright x = \omega^i x, \quad c^i \triangleright gx = \omega^i gx \tag{23}$$

for all $i = 0, 1, \cdots, n-1$. Following [1, Corollary 4.4] we denote by $H_{4n,\,\omega}$ the bicrossed product $H_4 \bowtie k[C_n]$ associated to this matched pair. In fact, as the right action $\triangleleft : k[C_n] \otimes H_4 \to k[C_n]$ is trivial, we have that $H_{4n,\,\omega} = H_4 \# k[C_n]$, the semi-direct product in the sense of Example 1.2. Thus, $H_{4n,\,\omega}$ is the quantum group at roots of unity generated as an algebra by g, x and c subject to the relations:

$$g^2 = c^n = 1, \quad x^2 = 0, \quad xg = -gx, \quad cg = gc, \quad cx = \omega\, xc$$

with the coalgebra structure given such that g and c are group-like elements and x is $(1, g)$-primitive. A k-basis in $H_{4n,\,\omega}$ is given by $\{c^i,\, gc^i,\, xc^i, gxc^i \mid i = 0, \cdots, n-1\}$. Using Corollary 2.10 we obtain:

Corollary 3.1. *Let k be a field of characteristic $\neq 2$ and ω a n-th root of unity in k. Then the canonical extension $H_4 \subseteq H_{4n,\,\omega}$ is rigid, that is $[H_{4n,\,\omega} : H_4]^f = 1$.*

Let ξ be a generator of the group $U_n(k)$. In what follows we will construct a family of matched pairs of Hopf algebras $(k[C_n],\, H_{4n,\,\xi^t},\, \triangleleft^l,\, \triangleright)$ such that the Hopf algebra $H_{4n,\,\xi^{t-lp}}$ will appear as an r-deformation (in the sense of Theorem 2.6) of $H_{4n,\,\xi^t}$.

Theorem 3.2. *Let k be a field of characteristic $\neq 2$, n a positive integer, ξ a generator of $U_n(k)$, $t \in \{0, 1, ..., \nu(n)-1\}$ and $C_n = \langle d \mid d^n = 1\rangle$ the cyclic group of order n. Then:*

(1) For any $l \in \{0, 1, \cdots, \nu(n) - 1\}$ there exists a matched pair $(k[C_n],\, H_{4n,\,\xi^t},\, \triangleleft^l,\, \triangleright)$, where $\triangleright : H_{4n,\,\xi^t} \otimes k[C_n] \to k[C_n]$ is the trivial action and the right action $\triangleleft^l : H_{4n,\,\xi^t} \otimes k[C_n] \to H_{4n,\,\xi^t}$ is given by:

$$c^i \triangleleft^l d^k = c^i, \quad (gc^i) \triangleleft^l d^k = gc^i, \quad (xc^i) \triangleleft^l d^k = \xi^{lk}\, xc^i, \quad (gxc^i) \triangleleft^l d^k = \xi^{lk}\, gxc^i \tag{24}$$

for all $i,\ k = 0, 1, ..., n-1$.

(2) The deformation maps associated to the matched pair $(k[C_n],\, H_{4n,\,\xi^t},\, \triangleleft^l,\, \triangleright)$ are the algebra maps defined as follows:

$$r_p : H_{4n,\,\xi^t} \to k[C_n], \quad r_p(g) = 1, \quad r_p(c) = d^p, \quad r_p(x) = 0$$

where $p \in \{0, 1, \cdots, n-1\}$. Furthermore, the r_p-deformation of $H_{4n,\,\xi^t}$ is $H_{4n,\,\xi^{t-lp}}$, i.e. $(H_{4n,\,\xi^t})_{r_p} = H_{4n,\,\xi^{t-lp}}$.

Proof. (1) First, notice that since $\triangleright$ is the trivial action and $k[C_n]$ is cocomutative then the compatibility condition (5) is trivially fulfilled. Moreover, (4) becomes:

$$(yz) \triangleleft^l a = (y \triangleleft^l a_{(1)})(z \triangleleft^l a_{(2)}) \tag{25}$$

for all $y,\, z \in H_{4n,\,\xi^t}$ and $a \in k[C_n]$. Since we have:

$$c^i \triangleleft^l d^k = c^i, \qquad g \triangleleft^l d^k = g, \qquad x \triangleleft^l d^k = \xi^{lk} x$$

then it is straightforward to see that (25) indeed holds. Now we prove that $\triangleleft^l : H_{4n,\,\xi^t} \otimes k[C_n] \to H_{4n,\,\xi^t}$ is a coalgebra map. Indeed, it is straightforward to see that $\Delta(c^i \triangleleft^l d^k) = c^i \triangleleft^l d^k \otimes c^i \triangleleft^l d^k$ and $\Delta(gc^i \triangleleft^l d^k) = gc^i \triangleleft^l d^k \otimes gc^i \triangleleft^l d^k$ for all $i,\ k \in 0, 1, ..., n-1$. Furthermore, we have:

$$\begin{aligned}
\Delta(xc^i \triangleleft^l d^k) &= (xc^i \triangleleft^l d^k)_{(1)} \otimes (xc^i \triangleleft^l d^k)_{(2)} \\
&= \xi^{lk}(xc^i)_{(1)} \otimes (xc^i)_{(2)} \\
&= \xi^{lk} xc^i \otimes c^i + \xi^{lk} gc^i \otimes xc^i \\
&= xc^i \triangleleft^l d^k \otimes c^i \triangleleft^l d^k + gc^i \triangleleft^l d^k \otimes xc^i \triangleleft^l d^k \\
&= (xc^i)_{(1)} \triangleleft^l d^k \otimes (xc^i)_{(2)} \triangleleft^l d^k
\end{aligned}$$

and

$$\begin{aligned}
\Delta(gxc^i \triangleleft^l d^k) &= (gxc^i \triangleleft^l d^k)_{(1)} \otimes (gxc^i \triangleleft^l d^k)_{(2)} \\
&= \xi^{lk}(gxc^i)_{(1)} \otimes (gxc^i)_{(2)} \\
&= \xi^{lk} gxc^i \otimes gc^i + \xi^{lk} c^i \otimes gxc^i \\
&= gxc^i \triangleleft^l d^k \otimes gc^i \triangleleft^l d^k + c^i \triangleleft^l d^k \otimes gxc^i \triangleleft^l d^k \\
&= (gxc^i)_{(1)} \triangleleft^l d^k \otimes (gxc^i)_{(2)} \triangleleft^l d^k
\end{aligned}$$

i.e. $\triangleleft^l : H_{4n,\,\xi^t} \otimes k[C_n] \to H_{4n,\,\xi^t}$ is a coalgebra map. Finally, we only need to prove that the action $\triangleleft^l$ respects the relations in $k[C_n]$, respectively $H_{4n,\,\xi^t}$. For instance, we have:

$$xc^i \triangleleft^l d^n = (xc^i \triangleleft^l d^{n-1}) \triangleleft^l d = \xi^{l(n-1)} xc^i \triangleleft^l d = \xi^{ln} xc^i = xc^i$$

$$xg \triangleleft^l d^k = (x \triangleleft^l d^k)(g \triangleleft^l d^k) = \xi^{lk} xg = -\xi^{lk} gx = -gx \triangleleft^l d^k$$

$$cx \triangleleft^l d^k = (c \triangleleft^l d^k)(x \triangleleft^l d^k) = \xi^{lk} cx = \xi^{lk} \xi^t xc = \xi^t xc \triangleleft^l d^k$$

Proving that the rest of the compatibilities also hold is a routinely check.

(2) Let $r : H_{4n,\,\xi^t} \to k[C_n]$ be a deformation map. By applying (11) for xc^i and gxc^i, where $i = 0, 1, \cdots, n-1$ we obtain:

$$r(c^i) \otimes xc^i + r(xc^i) \otimes gc^i = r(xc^i) \otimes c^i + r(gc^i) \otimes xc^i$$

$$r(gc^i) \otimes gxc^i + r(gxc^i) \otimes c^i = r(gxc^i) \otimes gc^i + r(c^i) \otimes gxc^i$$

Hence, it follows that $r(xc^i) = r(gxc^i) = 0$ and $r(c^i) = r(gc^i)$ for all $i \in 0, 1, ..., n-1$. In particular we have $r(g) = 1$ and $r(x) = 0$. Moreover, since r is also a coalgebra map then $r(c)$ is a grouplike element from $k[C_n]$. Consider $r(c) = d^p$, for some $p = 0, 1, \cdots, n-1$. For the rest of the proof we will denote this map by r_p.

As $\triangleright$ is the trivial action then the compatibility condition (12) simplifies to:

$$r_p\Big(\big(y \triangleleft^l r(z_{(1)})\big)\, z_{(2)}\Big) = r_p(y) r_p(z) \tag{26}$$

for all $y,\ z \in H_{4n,\,\xi^t}$. By applying (26) for c^i and c^j, where $i,\ j \in 0, 1, ..., n-1$ we get $r_p(c^{i+j}) = r_p(c^i) r_p(c^j)$. Hence, we have

$$r_p(c^i) = r_p(gc^i) = d^{ip} \tag{27}$$

for all $i = 0, 1, \cdots, n-1$. Now by using (27) and the fact that $r_p(xc^i) = r_p(gxc^i) = 0$, for any $i = 0, 1, \cdots, n-1$ we can easily prove that r_p is an algebra map. For instance, we have:

$$\begin{aligned}
r_p(c^i gxc^j) &= 0 = r_p(c^i) r_p(gxc^j) \\
r_p(xc^i c^j) &= 0 = r_p(xc^i) r_p(c^j) \\
r_p(xc^i xc^j) &= 0 = r_p(xc^i) r_p(xc^j) \\
r_p(c^i gc^j) &= r_p(gc^{i+j}) = d^{p(i+j)} = d^{pi} d^{pj} = r_p(c^i) r_p(gc^j) \\
r_p(gc^i gc^j) &= r_p(c^{i+j}) = d^{p(i+j)} = d^{pi} d^{pj} = r_p(gc^i) r_p(gc^j)
\end{aligned}$$

for all $i,\, j = 0, 1, \cdots, n-1$. It is straightforward to see that the rest of the compatibilities also hold and r_p is indeed an algebra map. Finally, we are left to prove that (26) holds. For instance we have:

$$\begin{aligned}
r_p\Big(\big(gc^i \triangleleft^l r_p(c^j)\big)\, c^j\Big) &= r_p\Big(\big(gc^i \triangleleft^l d^{pj}\big)\, c^j\Big) = r_p(gc^{i+j}) = r_p(gc^i) r_p(c^j)\\
r_p\Big(\big(xc^i \triangleleft^l r_p(c^j)\big)\, c^j\Big) &= r_p\Big(\big(xc^i \triangleleft^l d^{pj}\big)\, c^j\Big) = r_p(\xi^{lpj} xc^{i+j}) = 0 = r_p(xc^i) r_p(c^j)\\
r_p\Big(\big(y \triangleleft^l r_p((xc^i)_{(1)})\big)\, (xc^i)_{(2)}\Big) &= r_p\Big(\big(y \triangleleft^l r_p(xc^i)\big)\, c^i\Big) + r_p\Big(\big(y \triangleleft^l r_p(gc^i)\big)\, xc^i\Big)\\
&= 0 = r_p(y) r_p(xc^i)
\end{aligned}$$

for all $i,\, j = 0, 1, \cdots, n-1$ and $y \in H_{4n,\, \xi^t}$. By a straightforward computation it can be seen that (26) also holds for the remaining elements of the k-basis of $H_{4n,\, \xi^t}$.

Now, the algebra structure of $(H_{4n,\, \xi^t})_{r_p}$ is given by (14). Thus, in $(H_{4n,\, \xi^t})_{r_p}$ we have:

$$\begin{aligned}
g \bullet g &= \big(g \triangleleft^l r_p(g)\big) g = (g \triangleleft^l 1) g = g^2 = 1\\
x \bullet x &= (x \triangleleft^l r(x)) + (x \triangleleft^l r(g)) x = x^2 = 0\\
c^{n-1} \bullet c &= \big(c^{n-1} \triangleleft^l r_p(c)\big) c = \big(c^{n-1} \triangleleft^l c^p\big) c = c^{n-1} c = c^n = 1\\
g \bullet x &= (g \triangleleft^l r(x)) + (g \triangleleft^l r(g)) x = gx = -xg = -(x \triangleleft^l r(g)) g = -x \bullet g\\
c \bullet x &= (c \triangleleft^l r(x)) + (c \triangleleft^l r(g)) x = cx = \xi^t xc = \xi^{t-lp} (\xi^{lp} xc)\\
&= \xi^{t-lp} (x \triangleleft^l r_p(c)) c = \xi^{t-lp} x \bullet c
\end{aligned}$$

This shows that $(H_{4n,\, \xi^t})_{r_p} = H_{4n,\, \xi^{t-lp}}$. $\square$

The bicrossed product $k[C_n] \bowtie^l H_{4n,\, \xi^t}$ associated to the matched pair from Theorem 3.2 is the $4n^2$-dimensional Hopf algebra generated as an algebra by g, x, c and d subject to the relations

$$g^2 = c^n = d^n = 1, \quad x^2 = 0, \quad cg = gc, \quad cd = dc, \quad gd = dg,$$
$$xg = -gx, \quad cx = \xi^t\, xc, \quad xd = \xi^l\, dx$$

with the coalgebra structure given such that g, c, d are group-like elements and x is an $(1, g)$-primitive element. We denote by $H_{4n^2,\, \xi,\, t,\, l}$ this family of quantum groups, for any $l,\, t \in \{0, 1, ..., \nu(n) - 1\}$ and ξ a generator of order $\nu(n)$ of the group $U_n(k)$. In what follows we view $H_{4n^2,\, \xi,\, t,\, l}$ as a Hopf algebra extension of the group algebra $k[C_n] = k\langle d \,|\, d^n = 1\rangle$. In this context, $H_{4n,\, \xi^t}$ is a $k[C_n]$-complement of $H_{4n^2,\, \xi,\, t,\, l}$.

Before stating the main result of this section we recall from [1, Theorem 4.10] the number of types of isomorphisms of Hopf algebras $H_{4n,\, \omega}$, where $\omega \in U_n(k)$; we denote this number by $\# H_{4n,\, \omega}$. If $\nu(n) = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the prime decomposition of $\nu(n) = |U_n(k)|$ then:

$$\# H_{4n,\, \omega} = \left\{ \begin{array}{rll} (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1), & \text{if} & \nu(n) \ \text{is odd}\\ \alpha_1(\alpha_2 + 1) \cdots (\alpha_r + 1), & \text{if} & \nu(n) \ \text{is even and} \ p_1 = 2 \end{array} \right. \tag{28}$$

The main result of this section now follows: it computes the factorization index of the extension $k[C_n] \subseteq H_{4n^2,\, \xi,\, t,\, 1}$. This provides an extension whose factorization index is arbitrary large.

Theorem 3.3. *Let k be a field of characteristic $\neq 2$, n a positive integer, ξ a generator of $U_n(k)$ and $(k[C_n],\, H_{4n,\,\xi^{\nu(n)-1}},\, \triangleleft^1,\, \triangleright)$ the matched pair where $\triangleright$ is the trivial action and $\triangleleft^1$ is given by (24) for $l = 1$. Then:*

1) $(H_{4n,\,\xi^{\nu(n)-1}})_{r_p} = H_{4n,\,\xi^{\nu(n)-1-p}}$, for all $p = 0, 1, \cdots, \nu(n) - 1$. Thus, any $H_{4n,\,\xi^p}$ appears as a deformation of $H_{4n,\,\xi^{\nu(n)-1}}$, for some deformation map r_p.

2) Assume that $\nu(n)$ is odd and $\nu(n) = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the prime decomposition of $\nu(n)$. Then we have $(\alpha_1+1)(\alpha_2+1)...(\alpha_r+1)$ non-isomorphic deformations of $H_{4n,\,\xi^{\nu(n)-1}}$ and thus $[H_{4n^2,\,\xi,\,t,\,1} : k[C_n]]^f = (\alpha_1+1)(\alpha_2+1)...(\alpha_r+1)$.

3) Assume that $\nu(n)$ is even and $\nu(n) = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime decomposition of $\nu(n)$. Then we have $\alpha_1(\alpha_2+1)...(\alpha_r+1)$ non-isomorphic deformations of $H_{4n,\,\xi^{\nu(n)-1}}$ and thus $[H_{4n^2,\,\xi,\,t,\,1} : k[C_n]]^f = \alpha_1(\alpha_2+1)...(\alpha_r+1)$.

Proof. 1) It follows by applying Theorem 3.2 for $l = 1$ and $t = \nu(n) - 1$. As any $H_{4n,\,\xi^p}$ appears as a deformation of $H_{4n,\,\xi^{\nu(n)-1}}$ via some deformation map r_p, the last two statements are just easy consequences of (28). □

4. Classifying complements for Lie algebras

Throughout this section k is a field of characteristic zero. For a given algebra A we denote by A^L the vector space A with the Lie bracket $[x,\, y] := xy - yx$; $U(\mathfrak{g})$ will be the enveloping algebra of a Lie algebra $\mathfrak{g}$. In what follows, the canonical map $i_{\mathfrak{g}} : \mathfrak{g} \to U(\mathfrak{g})^L$ associated to a Lie algebra $\mathfrak{g}$ will be treated as an inclusion map. For two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ there exists a one to one correspondence between the set of all Lie algebra maps from $\mathfrak{g}$ to $\mathfrak{h}$ and the set of all Hopf algebra maps from $U(\mathfrak{g})$ and $U(\mathfrak{h})$. As a special case of [20, Theorem 15.3.4] we obtain that any Hopf subalgebra of an enveloping algebra $U(\mathfrak{g})$ is of the form $U(\mathfrak{f})$, for a Lie subalgebra $\mathfrak{f}$ of $\mathfrak{g}$.

Let Ξ be a Lie algebra and $\mathfrak{g} \subseteq \Xi$ be a given Lie subalgebra of Ξ. A Lie subalgebra $\mathfrak{h}$ of Ξ is called a *complement* of $\mathfrak{g}$ in Ξ (or a $\mathfrak{g}$-*complement* of Ξ) if every element of Ξ decomposes uniquely into an element of $\mathfrak{g}$ and an element of $\mathfrak{h}$, i.e. $\Xi = \mathfrak{g} + \mathfrak{h}$ and $\mathfrak{g} \cap \mathfrak{h} = \{0\}$. In this case we say that the Lie algebra Ξ factorizes through $\mathfrak{g}$ and $\mathfrak{h}$. Related to these concepts, by analogy with groups and Hopf algebras, the bicrossed product associated to a matched pairs of Lie algebras was introduced. We collect here some basic results about matched pairs and bicrossed products of Lie algebras. For all unexplained notations or definitions we refer the reader to [15], [16, Chapter 8] or [17]. A *matched pair of Lie algebras* [16, Definition 8.3.1] is a quadruple $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$, where $\mathfrak{g}$, $\mathfrak{h}$ are Lie algebras, $\mathfrak{g}$ is a left $\mathfrak{h}$-Lie module under $\triangleright : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}$, $\mathfrak{h}$ is a right $\mathfrak{g}$-Lie module under $\triangleleft : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{h}$ and the following compatibilities hold for all $a,\, b \in \mathfrak{g}$, $x,\, y \in \mathfrak{h}$:

$$x \triangleright [a,\, b] = [x \triangleright a,\, b] + [a,\, x \triangleright b] + (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a \tag{29}$$

$$[x,\, y] \triangleleft a = [x,\, y \triangleleft a] + [x \triangleleft a,\, y] + x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a) \tag{30}$$

The fact that $\mathfrak{g}$ is a left $\mathfrak{h}$-Lie module under $\triangleright : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{h}$ is a right $\mathfrak{g}$-Lie module under $\triangleleft : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{h}$ can be written explicitly as follows:

$$[x,\, y] \triangleright a = x \triangleright (y \triangleright a) - y \triangleright (x \triangleright a) \tag{31}$$

$$x \triangleleft [a,\, b] = (x \triangleleft a) \triangleleft b - (x \triangleleft b) \triangleleft a \tag{32}$$

The following is [16, Proposition 8.3.2]: If $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ is a matched pair of Lie algebras then the vector space $\mathfrak{g} \oplus \mathfrak{h}$ together with the bracket defined by:

$$[a \oplus x,\, b \oplus y] = \big([a,\, b] + x \triangleright b - y \triangleright a\big) \oplus \big([x,\, y] + x \triangleleft b - y \triangleleft a\big) \tag{33}$$

for all $a,\, b \in \mathfrak{g}$, $x,\, y \in \mathfrak{h}$ is a Lie algebra, called the *bicrossed product* of $\mathfrak{g}$ and $\mathfrak{h}$, and will be denoted by $\mathfrak{g} \bowtie \mathfrak{h}$. The Lie algebra $\mathfrak{h}$ is a complement of $\mathfrak{g}$ in the bicrossed product $\mathfrak{g} \bowtie \mathfrak{h}$. Conversely, if $\mathfrak{h}$ a complement of a Lie subalgebra $\mathfrak{g}$ in a Lie algebra Ξ, then there exists a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ such that the corresponding bicrossed product $\mathfrak{g} \bowtie \mathfrak{h}$ is isomorphic as a Lie algebra with Ξ. Moreover, the actions of the matched pair $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ are constructed from:

$$[x,\, a] = x \triangleright a \oplus x \triangleleft a \tag{34}$$

for all $a \in \mathfrak{g}$, $x \in \mathfrak{h}$. From now on, the matched pair constructed in (34) will be called *the canonical matched pair* associated to the factorization on Ξ through $\mathfrak{g}$ and $\mathfrak{h}$.

The connection between the bicrossed product of Hopf algebras and the bicrossed product of Lie algebras was proved in [17, Proposition 2.4]: if $\mathfrak{g}$ and $\mathfrak{h}$ are two Lie algebras, then there is a bijective correspondence between the matched pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ and the matched pairs of Hopf algebras $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$. The bijection is given such that the Hopf algebra $U(\mathfrak{g}) \bowtie U(\mathfrak{h})$ obtained from the matched pair $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g} \bowtie \mathfrak{h})$ of the Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$ arising from the matched pair $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$. For a future use we need the following:

Lemma 4.1. *Let $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ be the canonical matched pair of a factorization of a Lie algebra Ξ through $\mathfrak{g}$ and $\mathfrak{h}$ and $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ the corresponding matched pair of Hopf algebras. Then $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ is the canonical matched pair associated with the factorization of $U(\mathfrak{g}) \bowtie U(\mathfrak{h})$ through $U(\mathfrak{g})$ and $U(\mathfrak{h})$.*

Proof. We need to prove that:

$$xa = (x_{(1)} \triangleright a_{(1)})(x_{(2)} \triangleleft a_{(2)}) \tag{35}$$

holds for all $x \in \mathfrak{h}$ and $a \in \mathfrak{g}$. As x and a are primitive elements in the corresponding enveloping algebras, (35) is equivalent to:

$$xa = x \triangleright a + a \triangleleft x + ax \tag{36}$$

Since in $U(\mathfrak{g}) \bowtie U(\mathfrak{h}) \cong U(\mathfrak{g} \bowtie \mathfrak{h})$ we have $[x,\, a] = ax - xa$ it follows that (34) implies (36) and the proof is finished. $\square$

For a Lie subalgebra $\mathfrak{g}$ of Ξ we denote by $\mathcal{F}(\mathfrak{g}, \Xi)$ the isomorphism classes of all $\mathfrak{g}$-complements of Ξ. The *factorization index* of $\mathfrak{g}$ in Ξ is defined as $[\Xi : \mathfrak{g}]^f := |\, \mathcal{F}(\mathfrak{g}, \Xi)\, |$.

Definition 4.2. Let $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ be a matched pair of Lie algebras. A k-linear map $r : \mathfrak{h} \to \mathfrak{g}$ is called a *deformation map* of the matched pair $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ if the following compatibility holds for any $x,\, y \in \mathfrak{h}$:

$$r\big([x,\, y]\big) - \big[r(x),\, r(y)\big] = r\big(\, y \triangleleft r(x) - x \triangleleft r(y)\,\big) + x \triangleright r(y) - y \triangleright r(x) \tag{37}$$

We denote by $\mathcal{DM}\,(\mathfrak{h}, \mathfrak{g}\,|\,(\triangleright, \triangleleft))$ the set of all deformation maps of the matched pair $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$. The right hand side of (37) measures how far a deformation map is from being a Lie algebra map. The trivial map $r : \mathfrak{h} \to \mathfrak{g}$, $r(h) = 0$ is of course a deformation map. Using this concept a general deformation of a given Lie algebra is introduced in the next theorem:

Theorem 4.3. ***(Deformation of a Lie algebra)*** *Let $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ be a matched pair of Lie algebras and $r : \mathfrak{h} \to \mathfrak{g}$ a deformation map. Then $\mathfrak{h}_r := \mathfrak{h}$, as a k-space, with the new Lie bracket on $\mathfrak{h}$ defined by:*

$$[x,\, y]_r := [x,\, y] + x \triangleleft r(y) - y \triangleleft r(x) \tag{38}$$

for all $x,\, y \in \mathfrak{h}$ is a Lie algebra called the r-deformation of $\mathfrak{h}$.

Proof. Let $x,\, y,\, z \in \mathfrak{h}_r := \mathfrak{h}$. Then $[x,\, x]_r = x \triangleleft r(x) - x \triangleleft r(x) + [x,\, x] = 0$. On the other hand we have:

$$\begin{aligned}
& \big[x,\, \underline{[y,\, z]_r}\big]_r + \big[y,\, \underline{[z,\, x]_r}\big]_r + \big[z,\, \underline{[x,\, y]_r}\big]_r = \\
\overset{(38)}{=}\; & \big[x,\, y \triangleleft r(z) - z \triangleleft r(y) + [y,\, z]\big]_r + \big[y,\, z \triangleleft r(x) - x \triangleleft r(z) + \\
& [z,\, x]\big]_r + \big[z,\, x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y]\big]_r \\
\overset{(38)}{=}\; & x \triangleleft r\big(y \triangleleft r(z)\big) - \big(y \triangleleft r(z)\big) \triangleleft r(x) + [x,\, y \triangleleft r(z)] - x \triangleleft r\big(z \triangleleft r(y)\big) + \big(z \triangleleft r(y)\big) \triangleleft r(x) \\
& -[x,\, z \triangleleft r(y)] + \big(x \triangleleft r\big([y,\, z]\big)\big) - [y,\, z] \triangleleft r(x) + \underline{[x,\, [y,\, z]]} + y \triangleleft r\big(z \triangleleft r(x)\big) \\
& -\big(z \triangleleft r(x)\big) \triangleleft r(y) + [y,\, z \triangleleft r(x)] - y \triangleleft r\big(x \triangleleft r(z)\big) + \big(x \triangleleft r(z)\big) \triangleleft r(y) - [y,\, x \triangleleft r(z)] \\
& +\big(y \triangleleft r\big([z,\, x]\big)\big) - [z,\, x] \triangleleft r(y) + \underline{[y,\, [z,\, x]]} + z \triangleleft r\big(x \triangleleft r(y)\big) - \big(x \triangleleft r(y)\big) \triangleleft r(z) \\
& +\big[z,\, x \triangleleft r(y)\big] - z \triangleleft r\big(y \triangleleft r(x)\big) + \big(y \triangleleft r(x)\big) \triangleleft r(z) - [z,\, y \triangleleft r(x)] + z \triangleleft r\big([x,\, y]\big) \\
& -[x,\, y] \triangleleft r(z) + \underline{[z,\, [x,\, y]]} \\
=\; & x \triangleleft r\big(y \triangleleft r(z)\big) - \big(y \triangleleft r(z)\big) \triangleleft r(x) + [x,\, y \triangleleft r(z)] - x \triangleleft r\big(z \triangleleft r(y)\big) + \big(z \triangleleft r(y)\big) \triangleleft r(x) \\
& -[x,\, z \triangleleft r(y)] + \big(x \triangleleft r\big([y,\, z]\big)\big) - \underline{[y,\, z] \triangleleft r(x)} + y \triangleleft r\big(z \triangleleft r(x)\big) - \big(z \triangleleft r(x)\big) \triangleleft r(y) \\
& +[y,\, z \triangleleft r(x)] - y \triangleleft r\big(x \triangleleft r(z)\big) + \big(x \triangleleft r(z)\big) \triangleleft r(y) - [y,\, x \triangleleft r(z)] + \big(y \triangleleft r\big([z,\, x]\big)\big) \\
& -\underline{[z,\, x] \triangleleft r(y)} + z \triangleleft r\big(x \triangleleft r(y)\big) - \big(x \triangleleft r(y)\big) \triangleleft r(z) + \big[z,\, x \triangleleft r(y)\big] - z \triangleleft r\big(y \triangleleft r(x)\big) \\
& +\big(y \triangleleft r(x)\big) \triangleleft r(z) - [z,\, y \triangleleft r(x)] + z \triangleleft r\big([x,\, y]\big) - \underline{[x,\, y] \triangleleft r(z)} \\
\overset{(30)}{=}\; & x \triangleleft r\big(y \triangleleft r(z)\big) - \big(y \triangleleft r(z)\big) \triangleleft r(x) \underline{+ [x,\, y \triangleleft r(z)]} - x \triangleleft r\big(z \triangleleft r(y)\big) + \big(z \triangleleft r(y)\big) \triangleleft r(x) \\
& -[x,\, z \triangleleft r(y)] + x \triangleleft r\big([y,\, z]\big) - [y \triangleleft r(x),\, z] \underline{- [y,\, z \triangleleft r(x)]} - y \triangleleft \big(z \triangleright r(x)\big) \\
& +z \triangleleft \big(y \triangleright r(x)\big) + y \triangleleft r\big(z \triangleleft r(x)\big) - \big(z \triangleleft r(x)\big) \triangleleft r(y) \underline{+ [y,\, z \triangleleft r(x)]} - y \triangleleft r\big(x \triangleleft r(z)\big) \\
& +\big(x \triangleleft r(z)\big) \triangleleft r(y) - [y,\, x \triangleleft r(z)] + \big(y \triangleleft r\big([z,\, x]\big)\big) - [z \triangleleft r(y),\, x] \underline{- [z,\, x \triangleleft r(y)]} \\
& -z \triangleleft \big(x \triangleright r(y)\big) + x \triangleleft \big(z \triangleright r(y)\big) + z \triangleleft r\big(x \triangleleft r(y)\big) - \big(x \triangleleft r(y)\big) \triangleleft r(z) \underline{+ \big[z,\, x \triangleleft r(y)\big]} \\
& -z \triangleleft r\big(y \triangleleft r(x)\big) + \big(y \triangleleft r(x)\big) \triangleleft r(z) - [z,\, y \triangleleft r(x)] + z \triangleleft r\big([x,\, y]\big) - [x \triangleleft r(z),\, y] \\
& \underline{-[x,\, y \triangleleft r(z)]} - x \triangleleft \big(y \triangleright r(z)\big) + y \triangleleft \big(x \triangleright r(z)\big)
\end{aligned}$$

$$
\begin{aligned}
&= && x \triangleleft r\big(y \triangleleft r(z)\big)\underline{-\big(y \triangleleft r(z)\big)\triangleleft r(x)} - x \triangleleft r\big(z \triangleleft r(y)\big)\underline{+\big(z \triangleleft r(y)\big)\triangleleft r(x)} - [x,\, z \triangleleft r(y)]\\
&&& +\big(x \triangleleft r\big([y,\, z]\big)\big) - [y \triangleleft r(x),\, z] - y \triangleleft \big(z \triangleright r(x)\big) + z \triangleleft \big(y \triangleright r(x)\big) + y \triangleleft r\big(z \triangleleft r(x)\big)\\
&&& \underline{-\big(z \triangleleft r(x)\big)\triangleleft r(y)} - y \triangleleft r\big(x \triangleleft r(z)\big)\underline{+\big(x \triangleleft r(z)\big)\triangleleft r(y)} - [y,\, x \triangleleft r(z)] + \big(y \triangleleft r\big([z,\, x]\big)\big)\\
&&& -[z \triangleleft r(y),\, x] - z \triangleleft \big(x \triangleright r(y)\big) + x \triangleleft \big(z \triangleright r(y)\big) + z \triangleleft r\big(x \triangleleft r(y)\big)\underline{-\big(x \triangleleft r(y)\big)\triangleleft r(z)}\\
&&& -z \triangleleft r\big(y \triangleleft r(x)\big)\underline{+\big(y \triangleleft r(x)\big)\triangleleft r(z)} - [z,\, y \triangleleft r(x)] + z \triangleleft r\big([x,\, y]\big) - [x \triangleleft r(z),\, y]\\
&&& -x \triangleleft \big(y \triangleright r(z)\big) + y \triangleleft \big(x \triangleright r(z)\big)\\
&\overset{(32)}{=} && x \triangleleft r\big(y \triangleleft r(z)\big) + y \triangleleft \big[r(x),\, r(z)\big] - x \triangleleft r\big(z \triangleleft r(y)\big) + z \triangleleft \big[r(y),\, r(x)\big]\underline{-[x,\, z \triangleleft r(y)]}\\
&&& +\big(x \triangleleft r\big([y,\, z]\big)\big)\underline{-[y \triangleleft r(x),\, z]} - y \triangleleft \big(z \triangleright r(x)\big) + z \triangleleft \big(y \triangleright r(x)\big) + y \triangleleft r\big(z \triangleleft r(x)\big)\\
&&& -y \triangleleft r\big(x \triangleleft r(z)\big) + x \triangleleft \big[r(z),\, r(y)\big]\underline{-[y,\, x \triangleleft r(z)]} + \big(y \triangleleft r\big([z,\, x]\big)\big)\underline{-[z \triangleleft r(y),\, x]}\\
&&& -z \triangleleft \big(x \triangleright r(y)\big) + x \triangleleft \big(z \triangleright r(y)\big) + z \triangleleft r\big(x \triangleleft r(y)\big) - z \triangleleft r\big(y \triangleleft r(x)\big)\underline{-[z,\, y \triangleleft r(x)]}\\
&&& +z \triangleleft r\big([x,\, y]\big)\underline{-[x \triangleleft r(z),\, y]} - x \triangleleft \big(y \triangleright r(z)\big) + y \triangleleft \big(x \triangleright r(z)\big)\\
&= && x \triangleleft r\big(y \triangleleft r(z)\big) + y \triangleleft \big[r(x),\, r(z)\big] - x \triangleleft r\big(z \triangleleft r(y)\big) + z \triangleleft \big[r(y),\, r(x)\big] + \big(x \triangleleft r\big([y,\, z]\big)\big)\\
&&& -y \triangleleft \big(z \triangleright r(x)\big) + z \triangleleft \big(y \triangleright r(x)\big) + y \triangleleft r\big(z \triangleleft r(x)\big) - y \triangleleft r\big(x \triangleleft r(z)\big) + x \triangleleft \big[r(z),\, r(y)\big]\\
&&& +\big(y \triangleleft r\big([z,\, x]\big)\big) - z \triangleleft \big(x \triangleright r(y)\big) + x \triangleleft \big(z \triangleright r(y)\big) + z \triangleleft r\big(x \triangleleft r(y)\big) - z \triangleleft r\big(y \triangleleft r(x)\big)\\
&&& +z \triangleleft r\big([x,\, y]\big) - x \triangleleft \big(y \triangleright r(z)\big) + y \triangleleft \big(x \triangleright r(z)\big)\\
&= && x \triangleleft \Big(\underline{r\big(y \triangleleft r(z)\big) - r(z \triangleleft r(y)) + r\big([y,\, z]\big)} + \big[r(z),\, r(y)\big]\underline{+ z \triangleright r(y) - y \triangleright r(z)}\Big)\\
&&& y \triangleleft \Big(\big[r(x),\, r(z)\big]\underline{- z \triangleright r(x) + r\big(z \triangleleft r(x)\big) - r(x \triangleleft r(z)) + r\big([z,\, x]\big) + x \triangleright r(z)}\Big)\\
&&& z \triangleleft \Big(\big[r(y),\, r(x)\big]\underline{+ y \triangleright r(x) - x \triangleright r(y) + r\big(x \triangleleft r(y)\big) - r(y \triangleleft r(x)) + r\big([x,\, y]\big)}\Big)\\
&\overset{(37)}{=} && x \triangleleft \big(\big[r(z),\, r(y)\big] + \big[r(y),\, r(z)\big]\big) + y \triangleleft \big(\big[r(x),\, r(z)\big] + \big[r(z),\, r(x)\big]\big)\\
&&& + z \triangleleft \big(\big[r(y),\, r(x)\big] + \big[r(x),\, r(y)\big]\big) = 0
\end{aligned}
$$

where in the third equality as well as in the last one we applied Jacobi's identity for the bracket $[\,,\,]$. The proof is finished. $\square$

The Lie algebra version of Theorem 2.8 is the following:

Theorem 4.4. ***(Deformation of complements)*** *Let $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ be a matched pair of Lie algebras, $r : \mathfrak{h} \to \mathfrak{g}$ a deformation map and $\mathfrak{h}_r$ the r-deformation of $\mathfrak{h}$. We define a new Lie action $\triangleright^r : \mathfrak{h}_r \otimes \mathfrak{g} \to \mathfrak{g}$ given as follows for any $x \in \mathfrak{h}_r$ and $a \in \mathfrak{g}$:*

$$
x \triangleright^r a := \big[r(x),\, a\big] + x \triangleright a - r(x \triangleleft a) \tag{39}
$$

Then $(\mathfrak{g}, \mathfrak{h}_r, \triangleright^r, \triangleleft)$ is a matched pair of Lie algebras and $\mathfrak{h}_r$ is a $\mathfrak{g}$-complement of $\mathfrak{g} \bowtie \mathfrak{h}$.

Proof. We will first prove that $\triangleright^r : \mathfrak{h}_r \otimes \mathfrak{g} \to \mathfrak{g}$ is a left Lie action. For any $x, y \in \mathfrak{h}$ and $a \in \mathfrak{g}$, we have:

$$\begin{aligned}
& x \triangleright^r (y \triangleright^r a) - y \triangleright^r (x \triangleright^r a) = \\
\overset{(39)}{=}\ & \big[r(x),\, [r(y),\, a]\big] + \underline{x \triangleright [r(y),\, a]} - r\big(\underline{x \triangleleft [r(y),\, a]}\big) + [r(x),\, y \triangleright a] + x \triangleright (y \triangleright a) - \\
& r\big(x \triangleleft (y \triangleright a)\big) - \big[r(x),\, r(y \triangleleft a)\big] - x \triangleright r(y \triangleleft a) + r\big(x \triangleleft r(y \triangleleft a)\big) - \big[r(y),\, [r(x),\, a]\big] - \\
& \underline{y \triangleright [r(x),\, a]} + r\big(\underline{y \triangleleft [r(x),\, a]}\big) - [r(y),\, x \triangleright a] - y \triangleright (x \triangleright a) + r\big(y \triangleleft (x \triangleright a)\big) + \\
& [r(y),\, r(x \triangleleft a)] + y \triangleright r(x \triangleleft a) - r\big(y \triangleleft r(x \triangleleft a)\big) \\
\overset{(29),(32)}{=}\ & \big[r(x),\, [r(y),\, a]\big] + [x \triangleright r(y),\, a] \underline{+ [r(y),\, x \triangleright a]} + \big(x \triangleleft r(y)\big) \triangleright a - (x \triangleleft a) \triangleright r(y) - \\
& r\big((x \triangleleft r(y)) \triangleleft a\big) + r\big((x \triangleleft a) \triangleleft r(y)\big) \underline{+ [r(x),\, y \triangleright a]} + x \triangleright (y \triangleright a) - r\big(x \triangleleft (y \triangleright a)\big) - \\
& \big[r(x),\, r(y \triangleleft a)\big] - x \triangleright r(y \triangleleft a) + r\big(x \triangleleft r(y \triangleleft a)\big) - \big[r(y),\, [r(x),\, a]\big] - [y \triangleright r(x),\, a] \\
& \underline{- \big[r(x),\, y \triangleright a\big]} - \big(y \triangleleft r(x)\big) \triangleright a + (y \triangleleft a) \triangleright r(x) + r\big((y \triangleleft r(x)) \triangleleft a\big) - \\
& r\big((y \triangleleft a) \triangleleft r(x)\big) \underline{- [r(y),\, x \triangleright a]} - y \triangleright (x \triangleright a) + r\big(y \triangleleft (x \triangleright a)\big) + [r(y),\, r(x \triangleleft a)] \\
& + y \triangleright r(x \triangleleft a) - r\big(y \triangleleft r(x \triangleleft a)\big) \\
=\ & \underline{\big[r(x),\, [r(y),\, a]\big]} + [x \triangleright r(y),\, a] + \big(x \triangleleft r(y)\big) \triangleright a - (x \triangleleft a) \triangleright r(y) - r\big((x \triangleleft r(y)) \triangleleft a\big) \\
& + r\big((x \triangleleft a) \triangleleft r(y)\big) + x \triangleright (y \triangleright a) - r\big(x \triangleleft (y \triangleright a)\big) - \big[r(x),\, r(y \triangleleft a)\big] - x \triangleright r(y \triangleleft a) \\
& + r\big(x \triangleleft r(y \triangleleft a)\big) \underline{- \big[r(y),\, [r(x),\, a]\big]} - [y \triangleright r(x),\, a] - \big(y \triangleleft r(x)\big) \triangleright a + (y \triangleleft a) \triangleright r(x) \\
& + r\big((y \triangleleft r(x)) \triangleleft a\big) - r\big((y \triangleleft a) \triangleleft r(x)\big) - y \triangleright (x \triangleright a) + r\big(y \triangleleft (x \triangleright a)\big) + \\
& [r(y),\, r(x \triangleleft a)] + y \triangleright r(x \triangleleft a) - r\big(y \triangleleft r(x \triangleleft a)\big) \\
=\ & \big[[r(x),\, r(y)],\, a\big] + [x \triangleright r(y),\, a] + \big(x \triangleleft r(y)\big) \triangleright a \underline{- (x \triangleleft a) \triangleright r(y)} - r\big((x \triangleleft r(y)) \triangleleft a\big) \\
& \underline{+ r\big((x \triangleleft a) \triangleleft r(y)\big)} + x \triangleright (y \triangleright a) - r\big(x \triangleleft (y \triangleright a)\big) - \big[r(x),\, r(y \triangleleft a)\big] - x \triangleright r(y \triangleleft a) \\
& + r\big(x \triangleleft r(y \triangleleft a)\big) - [y \triangleright r(x),\, a] - \big(y \triangleleft r(x)\big) \triangleright a + (y \triangleleft a) \triangleright r(x) + r\big((y \triangleleft r(x)) \triangleleft a\big) \\
& - r\big((y \triangleleft a) \triangleleft r(x)\big) - y \triangleright (x \triangleright a) + r\big(y \triangleleft (x \triangleright a)\big) \underline{+ [r(y),\, r(x \triangleleft a)]} \\
& \underline{+ y \triangleright r(x \triangleleft a) - r\big(y \triangleleft r(x \triangleleft a)\big)} \\
\overset{(37)}{=}\ & \big[[r(x),\, r(y)],\, a\big] + [x \triangleright r(y),\, a] + \big(x \triangleleft r(y)\big) \triangleright a - r\big((x \triangleleft r(y)) \triangleleft a\big) + x \triangleright (y \triangleright a) \\
& - r\big(x \triangleleft (y \triangleright a)\big) \underline{- \big[r(x),\, r(y \triangleleft a)\big] - x \triangleright r(y \triangleleft a) + r\big(x \triangleleft r(y \triangleleft a)\big)} - [y \triangleright r(x),\, a] \\
& - \big(y \triangleleft r(x)\big) \triangleright a \underline{+ (y \triangleleft a) \triangleright r(x)} + r\big((y \triangleleft r(x)) \triangleleft a\big) \underline{- r\big((y \triangleleft a) \triangleleft r(x)\big)} - y \triangleright (x \triangleright a) \\
& + r\big(y \triangleleft (x \triangleright a)\big) - r\big([x \triangleleft a,\, y]\big) \\
\overset{(37)}{=}\ & \underline{\big[[r(x),\, r(y)],\, a\big] + [x \triangleright r(y),\, a]} + \big(x \triangleleft r(y)\big) \triangleright a - r\big((x \triangleleft r(y)) \triangleleft a\big) + x \triangleright (y \triangleright a) \\
& - r\big(x \triangleleft (y \triangleright a)\big) \underline{- [y \triangleright r(x),\, a]} - \big(y \triangleleft r(x)\big) \triangleright a + r\big((y \triangleleft r(x)) \triangleleft a\big) - y \triangleright (x \triangleright a) \\
& + r\big(y \triangleleft (x \triangleright a)\big) - r\big([x \triangleleft a,\, y]\big) - r\big([x,\, y \triangleleft a]\big)
\end{aligned}$$

$$\begin{aligned}
&\overset{(37)}{=} \big[r\big(x \triangleleft r(y)\big),\, a\big] - \big[r\big(y \triangleleft r(x)\big),\, a\big] + \big[r\big([x,\, y]\big),\, a\big] + \big(x \triangleleft r(y)\big) \triangleright a - r\big((x \triangleleft r(y)) \triangleleft a\big)\\
&\quad + x \triangleright (y \triangleright a) \underline{- r\big(x \triangleleft (y \triangleright a)\big)} - \big(y \triangleleft r(x)\big) \triangleright a + r\big((y \triangleleft r(x)) \triangleleft a\big) - y \triangleright (x \triangleright a)\\
&\quad \underline{+ r\big(y \triangleleft (x \triangleright a)\big) - r\big([x \triangleleft a,\, y]\big) - r\big([x,\, y \triangleleft a]\big)}\\
&\overset{(30)}{=} \big[r\big(x \triangleleft r(y)\big),\, a\big] - \big[r\big(y \triangleleft r(x)\big),\, a\big] + \big[r\big([x,\, y]\big),\, a\big] + \big(x \triangleleft r(y)\big) \triangleright a - r\big((x \triangleleft r(y)) \triangleleft a\big)\\
&\quad \underline{+ x \triangleright (y \triangleright a)} - \big(y \triangleleft r(x)\big) \triangleright a + r\big((y \triangleleft r(x)) \triangleleft a\big) \underline{- y \triangleright (x \triangleright a)} - r\big([x,\, y] \triangleleft a\big)\\
&\overset{(31)}{=} \big[r\big(x \triangleleft r(y)\big),\, a\big] - \big[r\big(y \triangleleft r(x)\big),\, a\big] + \big[r\big([x,\, y]\big),\, a\big] + \big(x \triangleleft r(y)\big) \triangleright a - r\big((x \triangleleft r(y)) \triangleleft a\big)\\
&\quad [x,\, y] \triangleright a - \big(y \triangleleft r(x)\big) \triangleright a + r\big((y \triangleleft r(x)) \triangleleft a\big) - r\big([x,\, y] \triangleleft a\big)\\
&= \big[r\big(x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y]\big),\, a\big] + \big(x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y]\big) \triangleright a\\
&\quad - r\Big(\big(x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y]\big) \triangleleft a\Big)\\
&\overset{(39)}{=} \big(\underline{x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y]}\big) \triangleright^r a \overset{(38)}{=} [x,\, y]_r \triangleright^r a
\end{aligned}$$

where in the fourth equality we used Jacobi's identity for the bracket $[\,,\,]$.

Next we prove that the compatibilities (29) and (30) hold true. Indeed, for all $a,\, b \in \mathfrak{g}$, $x,\, y \in \mathfrak{h}$ we have:

$$\begin{aligned}
&[x \triangleright^r a,\, b] + [a,\, x \triangleright^r b] + (x \triangleleft a) \triangleright^r b - (x \triangleleft b) \triangleright^r a =\\
&\overset{(39)}{=} \big[[r(x),\, a],\, b\big] + [x \triangleright a,\, b] \underline{- [r(x \triangleleft a),\, b]} + \big[a,\, [r(x),\, b]\big] + [a,\, x \triangleright b] \underline{- [a,\, r(x \triangleleft b)]}\\
&\quad \underline{+ [r(x \triangleleft a),\, b]} + (x \triangleleft a) \triangleright b - r\big((x \triangleleft a) \triangleleft b\big) \underline{- [r(x \triangleleft b),\, a]} - (x \triangleleft b) \triangleright a + r\big((x \triangleleft b) \triangleleft a\big)\\
&= \underline{\big[[r(x),\, a],\, b\big]} + [x \triangleright a,\, b] + \underline{\big[a,\, [r(x),\, b]\big]} + [a,\, x \triangleright b] + (x \triangleleft a) \triangleright b - r\big((x \triangleleft a) \triangleleft b\big)\\
&\quad - (x \triangleleft b) \triangleright a + r\big((x \triangleleft b) \triangleleft a\big)\\
&= \big[r(x),\, [a,\, b]\big] + \underline{[x \triangleright a,\, b] + [a,\, x \triangleright b] + (x \triangleleft a) \triangleright b} - r\big((x \triangleleft a) \triangleleft b\big) \underline{- (x \triangleleft b) \triangleright a}\\
&\quad + r\big((x \triangleleft b) \triangleleft a\big)\\
&\overset{(29)}{=} \big[r(x),\, [a,\, b]\big] + x \triangleright [a,\, b] \underline{- r\big((x \triangleleft a) \triangleleft b\big) + r\big((x \triangleleft b) \triangleleft a\big)}\\
&\overset{(32)}{=} \big[r(x),\, [a,\, b]\big] + x \triangleright [a,\, b] - r\big(x \triangleleft [a,\, b]\big) = x \triangleright^r [a,\, b]
\end{aligned}$$

thus (29) holds. Moreover, we have:

$$\begin{aligned}
&[x \triangleleft a,\, y]_r + [x,\, y \triangleleft a]_r + x \triangleleft (y \triangleright^r a) - y \triangleleft (x \triangleright^r a) =\\
&\overset{(38),(39)}{=} (x \triangleleft a) \triangleleft r(y) - y \triangleleft r(x \triangleleft a) + [x \triangleleft a,\, y] + x \triangleleft r(y \triangleleft a) - (y \triangleleft a) \triangleleft r(x) +\\
&\quad [x,\, y \triangleleft a] + x \triangleleft \big([r(y),\, a] + y \triangleright a - r(y \triangleleft a)\big) - y \triangleleft \big([r(x),\, a] + x \triangleright a - r(x \triangleleft a)\big)\\
&= (x \triangleleft a) \triangleleft r(y) \underline{- y \triangleleft r(x \triangleleft a)} + [x \triangleleft a,\, y] \underline{+ x \triangleleft r(y \triangleleft a)} - (y \triangleleft a) \triangleleft r(x) +\\
&\quad [x,\, y \triangleleft a] + x \triangleleft [r(y),\, a] + x \triangleleft (y \triangleright a) \underline{- x \triangleleft r(y \triangleleft a)} - y \triangleleft [r(x),\, a] - y \triangleleft (x \triangleright a)\\
&\quad \underline{+ y \triangleleft r(x \triangleleft a)}\big)
\end{aligned}$$

$$
\begin{aligned}
&= (x \triangleleft a) \triangleleft r(y) + [x \triangleleft a,\, y] - (y \triangleleft a) \triangleleft r(x) + [x,\, y \triangleleft a] + \underline{x \triangleleft [r(y),\, a]} + \\
&\quad x \triangleleft (y \triangleright a) - \underline{y \triangleleft [r(x),\, a]} - y \triangleleft (x \triangleright a) \\
&\overset{(32)}{=} \underline{(x \triangleleft a) \triangleleft r(y)} + [x \triangleleft a,\, y] \underline{- (y \triangleleft a) \triangleleft r(x)} + [x,\, y \triangleleft a] + \big(x \triangleleft r(y)\big) \triangleleft a \\
&\quad \underline{- (x \triangleleft a) \triangleleft r(y)} + x \triangleleft (y \triangleright a) - \big(y \triangleleft r(x)\big) \triangleleft a \underline{+ (y \triangleleft a) \triangleleft r(x)} - y \triangleleft (x \triangleright a) \\
&= \underline{[x \triangleleft a,\, y] + [x,\, y \triangleleft a]} + \big(x \triangleleft r(y)\big) \triangleleft a \underline{+ x \triangleleft (y \triangleright a)} - \big(y \triangleleft r(x)\big) \triangleleft a \underline{- y \triangleleft (x \triangleright a)} \\
&\overset{(30)}{=} \big(x \triangleleft r(y)\big) \triangleleft a - \big(y \triangleleft r(x)\big) \triangleleft a + [x,\, y] \triangleleft a \\
&= \big(\underline{x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y]}\big) \triangleleft a \overset{(38)}{=} [x,\, y]_r \triangleleft a
\end{aligned}
$$

Furthermore, if we denote by $\mathfrak{g} \bowtie^r \mathfrak{h}_r$ the bicrossed product corresponding to the new matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}_r, \triangleright^r, \triangleleft)$ then the map $\varphi : \mathfrak{g} \bowtie^r \mathfrak{h}_r \to \mathfrak{g} \bowtie \mathfrak{h}$ defined by:

$$
\varphi(a \oplus x) := \big(a + r(x)\big) \oplus x
$$

for all $a \in \mathfrak{g}$, $x \in \mathfrak{h}_r = \mathfrak{h}$ is an isomorphism of Lie algebras with the inverse $\psi : \mathfrak{g} \bowtie \mathfrak{h} \to \mathfrak{g} \bowtie^r \mathfrak{h}_r$ given by $\psi(a \oplus x) := \big(a - r(x)\big) \oplus x$. Indeed, for all $a,\, b \in \mathfrak{g}$, $x,\, y \in \mathfrak{h}$ we have:

$$
\begin{aligned}
&\varphi\big([a \oplus x,\, b \oplus y]\big) \overset{(33)}{=} \varphi\big([a,\, b] + x \triangleright^r b - y \triangleright^r a \oplus [x,\, y]_r + x \triangleleft b - y \triangleleft a\big) \\
&= \Big([a,\, b] + \underline{x \triangleright^r b} - \underline{y \triangleright^r a} + r\big([x,\, y]_r + x \triangleleft b - y \triangleleft a\big)\Big) \oplus \big(\underline{[x,\, y]_r} + x \triangleleft b - y \triangleleft a\big) \\
&\overset{(38),(39)}{=} \Big([a,\, b] + [r(x),\, b] + x \triangleright b - r(x \triangleleft b) - [r(y),\, a] - y \triangleright a + r(y \triangleleft a) + \underline{r\big([x,\, y]_r\big)} \\
&\quad + r(x \triangleleft b) - r(y \triangleleft a)\Big) \oplus \big(x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y] + x \triangleleft b - y \triangleleft a\big) \\
&\overset{(37),(38)}{=} \Big([a,\, b] + [r(x),\, b] + x \triangleright b - [r(y),\, a] - y \triangleright a + \big[r(x),\, r(y)\big] + x \triangleright r(y) - y \triangleright r(x)\Big) \\
&\quad \oplus \big(x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y] + x \triangleleft b - y \triangleleft a\big) \\
&= \Big(\big[a + r(x),\, b + r(y)\big] + x \triangleright \big(b + r(y)\big) - y \triangleright \big(a + r(x)\big)\Big) \oplus \Big([x,\, y] + x \triangleleft \big(b + r(y)\big) \\
&\quad - y \triangleleft \big(a + r(x)\big)\Big) \overset{(33)}{=} \big[\big(a + r(x)\big) \oplus x,\, \big(b + r(y)\big) \oplus y\big] = \big[\varphi(a \oplus x),\, \varphi(b \oplus y)\big]
\end{aligned}
$$

Thus we proved that φ is a Lie algebra isomorphism. Moreover, $\mathfrak{g} \cong \varphi(\mathfrak{g} \oplus 0) = \mathfrak{g} \oplus 0$ and $\mathfrak{h}_r \cong \varphi(0 \oplus \mathfrak{h}) = \{\big(r(x) \oplus x\big) \mid x \in \mathfrak{h}\}$ are Lie subalgebras of $\mathfrak{g} \bowtie \mathfrak{h}$:

$$
\begin{aligned}
\big[r(x) \oplus x,\, r(y) \oplus y\big] &\overset{(33)}{=} \big(\underline{\big[r(x),\, r(y)\big] + x \triangleright r(y) - y \triangleright r(x)}\big) \oplus \big([x,\, y] + x \triangleleft r(y) - y \triangleleft r(x)\big) \\
&\overset{(37)}{=} r\big([x,\, y] + x \triangleleft r(y) - y \triangleleft r(x)\big) \oplus \big([x,\, y] + x \triangleleft r(y) - y \triangleleft r(x)\big)
\end{aligned}
$$

Therefore as $\mathfrak{g} \oplus 0$ and $\{\big(r(x) \oplus x\big) \mid x \in \mathfrak{h}\}$ have trivial intersection and

$$
a \oplus x = \Big(\big(a - r(x)\big) \oplus 0\Big) + \big(r(x) \oplus x\big)
$$

for all $a \in \mathfrak{g}$, $x \in \mathfrak{h}$ we can conclude that $\mathfrak{h}_r$ is a $\mathfrak{g}$-complement of $\mathfrak{g} \bowtie \mathfrak{h}$. $\square$

Proposition 4.5. *Let $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ be a matched pair of Lie algebras. Then any deformation map $\widehat{r} : U(\mathfrak{h}) \to U(\mathfrak{g})$ of the corresponding matched pair of Hopf algebras $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ induces a unique deformation map $r : \mathfrak{h} \to \mathfrak{g}$ of the matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$.*

Proof. As $\widehat{r} : U(\mathfrak{h}) \to U(\mathfrak{g})$ is a coalgebra map we have $\widehat{r}_{|P(U(\mathfrak{h}))} \subseteq P(U(\mathfrak{g}))$, where $P(H)$ denotes the set of primitive elements of a Hopf algebra H. Since the characteristic of k is zero we have $P(U(\mathfrak{g})) = \mathfrak{g}$. Therefore, we obtain a k-linear map $r : \mathfrak{h} \to \mathfrak{g}$, where $r = \widehat{r}_{|\mathfrak{g}}$. We are left to prove that (37) holds. Indeed, as $\widehat{r}$ is a deformation map of the matched pair $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ then it follows from (12) that for all x, $y \in \mathfrak{h}$ we have:

$$r(x \triangleleft r(y)) + \widehat{r}(xy) = r(x)r(y) + x \triangleright r(y)$$
$$r(y \triangleleft r(x)) + \widehat{r}(yx) = r(y)r(x) + y \triangleright r(x)$$

By subtracting the two equalities above it follows that (37) indeed holds. □

The next result proves that the deformation of an enveloping algebra in the sense of Theorem 2.6 is isomorphic to the enveloping algebra of a deformation of a Lie algebra in the sense of Theorem 4.3.

Proposition 4.6. *Let $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ be a matched pair of Lie algebras. If $\widehat{r} : U(\mathfrak{h}) \to U(\mathfrak{g})$ is a deformation map of the corresponding matched pair of Hopf algebras $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ and $r : \mathfrak{g} \to \mathfrak{h}$ the induced deformation map of the matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$, then we have an isomorphism of Hopf algebras $U(\mathfrak{h}_r) \simeq U(\mathfrak{h})_{\widehat{r}}$.*

Proof. To start with, we should notice that since $\mathfrak{h}_r = \mathfrak{h}$ as k-spaces we can consider the inclusion map $\bar{i} : \mathfrak{h}_r \to U(\mathfrak{h})^L_{\widehat{r}}$. We will prove that $\bar{i}$ is a Lie algebra map. Recall that the bracket $\overline{[\,,\,]}$ on the Lie algebra $U(\mathfrak{h})^L_{\widehat{r}}$ is given as follows for all x, $y \in \mathfrak{h}$:

$$\overline{[x,\, y]} := x \bullet y - y \bullet x \stackrel{(14)}{=} x \widehat{\triangleleft} \widehat{r}(y) + xy - y \widehat{\triangleleft} \widehat{r}(x) - yx = x \triangleleft r(y) + xy - y \triangleleft r(x) - yx$$

Therefore, we have:

$$\overline{[\bar{i}(x),\, \bar{i}(y)]} = \overline{[x,\, y]} = x \triangleleft r(y) + xy - y \triangleleft r(x) - yx$$

$$\bar{i}([x,\, y]_r) = \bar{i}\big(x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y]\big) = x \triangleleft r(y) - y \triangleleft r(x) + [x,\, y]$$

for all x, $y \in \mathfrak{h}$. Since in $U(\mathfrak{h})^L_{\widehat{r}}$ we have $[x,\, y] = xy - yx$ for all x, $y \in \mathfrak{h}$ it follows that $\overline{[\bar{i}(x),\, \bar{i}(y)]} = \bar{i}([x,\, y]_r)$ and thus $\bar{i}$ is indeed a Lie algebra map.

By the universal property of the universal enveloping algebra $U(\mathfrak{h}_r)$ we obtain an unique morphism of algebras $F : U(\mathfrak{h}_r) \to U(\mathfrak{h})_{\widehat{r}}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{h}_r \arrow[r, "i_{\mathfrak{h}_r}"] \arrow[dr, "\bar{i}"'] & U(\mathfrak{h}_r)^L \arrow[d, "F"] \\
& U(\mathfrak{h})^L_{\widehat{r}}
\end{tikzcd}$$

i.e. $F \circ i_{\mathfrak{h}_r} = \bar{i}$. Thus $F_{|\mathfrak{h}_r} = \mathrm{Id}_{|\mathfrak{h}_r}$. As $U(\mathfrak{h}_r)^L$ and $U(\mathfrak{h})^L_{\widehat{r}}$ are generated as algebras by $\mathfrak{h}_r$ and respectively $\mathfrak{h}$ it is straightforward to see that F is an isomorphism and also a coalgebra map. This proves that $U(\mathfrak{h}_r)$ and $U(\mathfrak{h})_{\widehat{r}}$ are isomorphic Hopf algebras. □

We are now able to describe the complements of a Lie subalgebra $\mathfrak{g}$ of Ξ:

Theorem 4.7. ***(Description of complements)*** *Let $\mathfrak{g}$ be a Lie subalgebra of Ξ, $\mathfrak{h}$ a given $\mathfrak{g}$-complement of Ξ with the associated canonical matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$. Then $\overline{\mathfrak{h}}$ is a $\mathfrak{g}$-complement of Ξ if and only if there exists an isomorphism of Lie algebras $\overline{\mathfrak{h}} \cong \mathfrak{h}_r$, for some deformation map $r : \mathfrak{h} \to \mathfrak{g}$ of the canonical matched pair $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$.*

Proof. Let $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ be the matched pair of Hopf algebras corresponding to the matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ such that the Hopf algebra $U(\mathfrak{g}) \bowtie U(\mathfrak{h})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g} \bowtie \mathfrak{h})$. Then, by Lemma 4.1 it follows that $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ is the canonical matched pair associated with the factorization of $U(\mathfrak{g}) \bowtie U(\mathfrak{h})$ through $U(\mathfrak{g})$ and $U(\mathfrak{h})$.

Let $(\mathfrak{g}, \overline{\mathfrak{h}}, \overline{\triangleright}, \overline{\triangleleft})$ be the canonical matched pair associated with the factorization of Ξ through $\mathfrak{g}$ and $\overline{\mathfrak{h}}$, and consider $(U(\mathfrak{g}), U(\overline{\mathfrak{h}}), \widehat{\overline{\triangleright}}, \widehat{\overline{\triangleleft}})$ to be the corresponding matched pair of Hopf algebras. By Theorem 2.9, there exists a deformation map $\widehat{r} : U(\mathfrak{h}) \to U(\mathfrak{g})$ of the matched pair of Hopf algebras $(U(\mathfrak{g}), U(\mathfrak{h}), \widehat{\triangleright}, \widehat{\triangleleft})$ such that we have an isomorphism of Hopf algebras $U(\overline{\mathfrak{h}}) \cong U(\mathfrak{h})_{\widehat{r}}$, $\widehat{\overline{\triangleright}} = \widehat{\triangleright}^{\widehat{r}}$ and $\widehat{\overline{\triangleleft}} = \widehat{\triangleleft}$. Moreover, by Proposition 4.6 we have an isomorphism of Hopf algebras $U(\mathfrak{h})_{\widehat{r}} \cong U(\mathfrak{h}_r)$, where $r : \mathfrak{h} \to \mathfrak{g}$ is the deformation map of the matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ induced by $\widehat{r} : U(\mathfrak{h}) \to U(\mathfrak{g})$. Therefore, the matched pair of Hopf algebras $(U(\mathfrak{g}), U(\overline{\mathfrak{h}}), \widehat{\overline{\triangleright}}, \widehat{\overline{\triangleleft}})$ is in fact of the form $(U(\mathfrak{g}), U(\mathfrak{h}_r), \widehat{\triangleright}^{\widehat{r}}, \widehat{\triangleleft})$. Thus we obtain that the corresponding matched pair of Lie algebras $(\mathfrak{g}, \overline{\mathfrak{h}}, \overline{\triangleright}, \overline{\triangleleft})$ is of the form $(\mathfrak{g}, \mathfrak{h}_r, \triangleright^r, \triangleleft)$. Hence we have an isomorphism of Lie algebras $\overline{\mathfrak{h}} \cong \mathfrak{h}_r$, where $r : \mathfrak{h} \to \mathfrak{g}$ is a deformation map of the matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$. □

In order to give the answer to the (CCP) for Lie algebras we need to introduce the following:

Definition 4.8. Let $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ be a matched pair of Lie algebras. Two deformation maps r, $R : \mathfrak{h} \to \mathfrak{g}$ are called *equivalent* and we denote this by $r \sim R$ if there exists $\sigma : \mathfrak{h} \to \mathfrak{h}$ a k-linear automorphism of $\mathfrak{h}$ such that:

$$\sigma\big([x,\, y]\big) - \big[\sigma(x),\, \sigma(y)\big] = \sigma(x) \triangleleft R\big(\sigma(x)\big) - \sigma\big(x \triangleleft r(y)\big) - \sigma(y) \triangleleft R\big(\sigma(x)\big) + \sigma\big(y \triangleleft r(x)\big) \tag{40}$$

for all x, $y \in \mathfrak{h}$.

Now, as a conclusion of all the above results we can prove the main result of this section:

Theorem 4.9. ***(Classification of complements)*** *Let $\mathfrak{g}$ be a Lie subalgebra of Ξ, $\mathfrak{h}$ a $\mathfrak{g}$-complement of Ξ and $(\mathfrak{g}, \mathfrak{h}, \triangleright, \triangleleft)$ the associated canonical matched pair. Then:*

(1) *$\sim$ is an equivalence relation on $\mathcal{DM}\,(\mathfrak{h}, \mathfrak{g} \,|\, (\triangleright, \triangleleft))$. We denote by $\mathcal{HA}^2(\mathfrak{h}, \mathfrak{g} \,|\, (\triangleright, \triangleleft))$ the quotient set $\mathcal{DM}\,(\mathfrak{h}, \mathfrak{g} \,|\, (\triangleright, \triangleleft))/\sim$.*

(2) *There exists a bijection between the isomorphism classes of all $\mathfrak{g}$-complements of Ξ and $\mathcal{HA}^2(\mathfrak{h}, \mathfrak{g} \,|\, (\triangleright, \triangleleft))$. In particular, the factorization index of $\mathfrak{g}$ in Ξ is computed by the formula:*

$$[\Xi : \mathfrak{g}]^f = |\mathcal{HA}^2(\mathfrak{h}, \mathfrak{g} \,|\, (\triangleright, \triangleleft))|$$

Proof. We proceed as in the proof of Theorem 2.5. It follows from Theorem 4.7 that if $\overline{\mathfrak{h}}$ is an arbitrary $\mathfrak{g}$-complement of Ξ, then there exists an isomorphism of Lie algebras $\overline{\mathfrak{h}} \cong \mathfrak{h}_r$, for some deformation map $r : \mathfrak{h} \to \mathfrak{g}$. Thus, in order to classify all $\mathfrak{g}$-complements of Ξ it is enough to consider only r-deformations of $\mathfrak{h}$, for various deformation maps $r : \mathfrak{h} \to \mathfrak{g}$. Now let r, $R : \mathfrak{h} \to \mathfrak{g}$ be two deformation maps. As $\mathfrak{h}_r = \mathfrak{h}_R := \mathfrak{h}$ as k-spaces, we obtain that the Lie algebras $\mathfrak{h}_r$ and $\mathfrak{h}_R$ are isomorphic if and only if there exists $\sigma : \mathfrak{h}_r \to \mathfrak{h}_R$ a k-linear isomorphism which is also a Lie algebra map. Using (38) it is straightforward to see that σ is a Lie algebra map if and only if the compatibility condition (40) of Definition 4.8 holds, i.e. $r \sim R$.

Hence, $r \sim R$ if and only if $\sigma : \mathfrak{h}_r \to \mathfrak{h}_R$ is an isomorphism of Lie algebras. Thus we obtain that $\sim$ is an equivalence relation on $\mathcal{DM}(\mathfrak{h}, \mathfrak{g} \,|\, (\triangleright, \triangleleft))$ and the map

$$\mathcal{HA}^2(\mathfrak{h}, \mathfrak{g} \,|\, (\triangleright, \triangleleft)) \to \mathcal{F}(\mathfrak{g}, \Xi), \qquad \overline{r} \mapsto \mathfrak{h}_r$$

where $\overline{r}$ is the equivalence class of r via the relation $\sim$, is well defined and a bijection between sets. □

5. Outlooks and open problems

In this paper we solve the problem of classifying all complements of a subobject A of E for the category of Lie algebras and Hopf algebras. The common tool used is the bicrossed product introduced by Takeuchi for groups and generalized by Majid for Hopf algebras and Lie algebras. These cases were also the source of inspiration for the introduction of the bicrossed product for other fields of mathematics like: algebras, C^*-algebras or von Neumann algebras, coalgebras, Lie groups, locally compact groups or locally compact quantum groups, groupoids or quantum grupoids, multiplier Hopf algebras, etc. Thus, all the results proven in this paper can serve as a model for obtaining similar answers for the (CCP) problem in all the fields mentioned above. Another direction for further study is given by the following two open questions:

Question 1: *Let $\sigma : H \otimes H \to H \to k$ be a normalized Sweedler cocycle and H_σ be Doi's [8] deformation of the Hopf algebra H. Does there exist a Hopf algebra A, a matched pair of Hopf algebras $(A, H, \triangleleft, \triangleright)$ and a deformation map $r : H \to A$ such that $H_\sigma = H_r$, where H_r is the r-deformation of H in the sense of Theorem 2.6?*

Having in mind Theorem 2.5 it is natural to ask:

Question 2: *Does there exist an example of an extension of finite dimensional Hopf algebras $A \subset E$ having an infinite factorization index $[E : A]^f$?*

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Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

E-mail address: `ana.agore@vub.ac.be` and `ana.agore@gmail.com`

Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania

E-mail address: `gigel.militaru@fmi.unibuc.ro` and `gigel.militaru@gmail.com`